Asset Pricing Theory: S05

Asset Pricing with Frictions

Amedeo Andriollo

Finance Department (DRM)
Université Paris Dauphine – PSL.

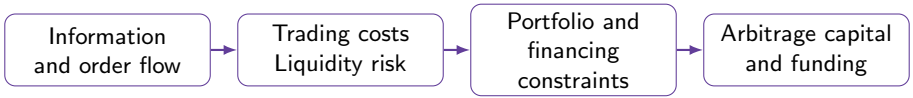
Structure of the course

- Email: amedeo.andriollo@dauphine.psl.eu
- Grading follows a 30-70 rule: 70% final exam, 30% project/homework.
- Office hours: feel "free" to (DO) come: B612.
- Slides/materials are updated regularly. Report a typo/mistake: corrections help the class.
- This session:
  - Information and price discovery, trading costs and liquidity risk, portfolio and financing constraints, and limits to arbitrage.
  - Main references: Glosten-Milgrom (1985); Kyle (1985); Acharya-Pedersen (2005); Frazzini-Pedersen (2014); Shleifer–Vishny (1997).

▷ Information and Asset Pricing.

Today's Route: When Trade Is Imperfect

- **Sessions 1–2:** NA and household optimality. **Session 3:** Intertemporal optimality. **Session 4:** Firms' investment decision.
- **This session:** *Who can trade claims, with what information, at what cost, and with what financing?*



- (Frictionless) Benchmark: Investors can adjust positions in either direction at a single price, w/out trading costs or binding financing constraints.

↪ **With frictions:** Trading may reveal information, execution may be costly, desired positions may be infeasible, and arbitrage may be limited.

▷ For each friction, we will identify the affected marginal decision, derive the relevant optimality condition, and examine its pricing implications.

The Frictionless Benchmark

- **Sessions 1–2:** An investor can buy/sell an additional share at the same market price $P_{i,t}$. The share delivers tomorrow $X_{i,t+1} = P_{i,t+1} + D_{i,t+1}$, so: $P_{i,t} = \mathbb{E}_t \left[M_{t+1} X_{i,t+1} \right], \quad M_{t+1} = \beta u_{C,t+1} / u_{C,t}$.
- **Session 4:** The firm faces the same intertemporal trade-off when choosing physical investment:

$$q_t = \mathbb{E}_t \left[M_{t+1} \left(F_{K,t+1} - G_{K,t+1} + (1-\delta) q_{t+1} \right) \right].$$

↪ The investor/firm can adjust the position in either direction.

- Financial trades occur at a **single price, w/out transaction costs or portfolio constraints**.
- Replicating a payoff additionally requires that the necessary portfolio can be constructed and financed.

- With frictions we may add different dimensions:
  - **Information:** Traders may have different information; order flow reveals information and affects quotes.
  - **Trading costs:** Buying/selling can generate different executable cash flows.
  - **Portfolio constraints:** Some marginal trades are infeasible (moving from equalities to inequalities).
  - **Scarce funding:** Financing a position carries an additional cost.

↪ A friction can change the price *w/out changing its underlying payoff*.

⊵ What changes is the ability/cost/info available to trade.

Glosten-Milgrom (1/4): Learning from Trade

- *Can trading change asset prices when the fundamental value has not changed?*
  Shock to fundamental changes the payoffs; informative trade changes what mkt participants **know** about them.
- Consider an asset with uncertain terminal value: $V \in \{V_L, V_H\}, \quad V_H > V_L, \quad q = \Pr(V = V_H) \in (0,1)$.
  Nature draws first V and then a trader's type.

  A competitive risk-neutral dealer posts an ask and a bid *before observing the order*.
  - **Informed** trader (probability $\alpha \in [0,1)$): observes V and buys one share if $V = V_H$, sells one if $V = V_L$.
  - **Uninformed** trader (probability $1-\alpha$): buys/sells one share with prob. $1/2$ each, independently of V.
  - The **dealer** observes the **direction of the order**, neither the trader's type nor V.
  - **Ask**: lowest price a seller is willing to trade VS. **Bid**: the highest price a buyer is willing to pay.
- Denote B buy and S sell:

	$V = V_H$	$V = V_L$
$\Pr(B \mid V)$	$\alpha + \frac{1-\alpha}{2} = \frac{1+\alpha}{2}$	$\frac{1-\alpha}{2}$
$\Pr(S \mid V)$	$\frac{1-\alpha}{2}$	$\alpha + \frac{1-\alpha}{2} = \frac{1+\alpha}{2}$

↪ **Economic interpretation:** When $\alpha > 0$, a buy is more likely if $V = V_H$, while a sell is more likely if $V = V_L$. Uninformed orders ($\alpha \to 0$) prevent the dealer from inferring V perfectly.

▷ **Adverse selection:** The dealer faces different expected asset values conditional on buying versus selling.

Glosten-Milgrom (2/4): Zero Dealer Profit

- Let $\Delta V = V_H - V_L$ then: (Bayes')

$$q_B \equiv \Pr(V_H \mid B) = \frac{\Pr(B \mid V_H)q}{\Pr(B)} = \frac{q(1+\alpha)}{q(1+\alpha)+(1-q)(1-\alpha)},$$

$$q_S \equiv \Pr(V_H \mid S) = \frac{\Pr(S \mid V_H)q}{\Pr(S)} = \frac{q(1-\alpha)}{q(1-\alpha)+(1-q)(1+\alpha)}.$$

  ↪ For $0 < \alpha < 1$, we have $q_B > q > q_S$: buying raises the posterior probability of a high fundamental value; selling lowers it.

- The dealer is risk-neutral and earns zero expected profit *conditional on each executed trade*:

$$\underbrace{\mathbb{E}[Ask - V \mid B] = 0}_{\text{dealer sells at the ask}} \implies \boxed{Ask = \mathbb{E}[V \mid B] = V_L + q_B \Delta V},$$

$$\underbrace{\mathbb{E}[V - Bid \mid S] = 0}_{\text{dealer buys at the bid}} \implies \boxed{Bid = \mathbb{E}[V \mid S] = V_L + q_S \Delta V}.$$

- **Why is there a spread?** Since $q_S < q < q_B$: $Bid < \mathbb{E}[V] = V_L + q\Delta V < Ask$, $\quad Ask - Bid = (q_B - q_S)\Delta V > 0$.
  ↪ A buy (sell) signals that V is more likely to be high (sell). The dealer charges a higher ask (lower bid) to avoid systematically selling undervalued (buying overvalued) shares.

↪ **Adverse selection:** Informed traders profit by buying when $V = V_H$ and selling when $V = V_L$.
The dealer offsets losses through gains on uninformed orders, breaking even on average.

▷ A positive bid–ask spread can therefore arise purely from asymmetric information.

Glosten-Milgrom (3/4): The Spread

- **Symmetric prior:** Suppose $q = \Pr(V_H) = 1/2$, so $\Pr(B) = \Pr(S) = 1/2$. By Bayes' rule:

$$q_B = \frac{1+\alpha}{2}, \qquad q_S = \frac{1-\alpha}{2}.$$

Denoting $\bar{V} = (V_H + V_L)/2$, the competitive quotes become:

$$Ask = \bar{V} + \frac{\alpha \Delta V}{2}, \qquad Bid = \bar{V} - \frac{\alpha \Delta V}{2}.$$

Thus, the **bid–ask spread** is:

$$\boxed{Ask - Bid = \alpha \Delta V = 2\alpha \sigma_V}, \qquad \sigma_V = \frac{\Delta V}{2}.$$

↪ $\alpha \uparrow$: More informed trading makes each order more informative, widening the spread.
↪ $\Delta V \uparrow$: A larger fundamental value gap increases the dealer's potential loss to informed traders.

! **Uncertainty alone is not sufficient:** If $\alpha = 0$, then $q_B = q_S = q$ and $Ask = Bid = \mathbb{E}[V]$, even when $\sigma_V > 0$. Without informed traders, orders reveal no information about V.

≥ The spread depends on both **fundamental uncertainty** (ΔV) and **asymmetric information** (α).

Glosten-Milgrom (4/4): Beyond Symmetry

- Let $q = \Pr(V_H) \in (0,1)$. By the law of total probability:

$$\Pr(B) = q\frac{1+\alpha}{2} + (1-q)\frac{1-\alpha}{2} = \frac{1+\alpha(2q-1)}{2}, \qquad \Pr(S) = \frac{1-\alpha(2q-1)}{2}.$$

  $\hookrightarrow$ For $\alpha > 0$ and $q \neq 1/2$, buys and sells are no longer equally likely.

- **Bayesian posteriors:** Define $d = \alpha(2q-1)$. Using the posteriors derived previously:

$$q_B - q_S = q\left(\frac{1+\alpha}{1+d} - \frac{1-\alpha}{1-d}\right) = \frac{q[(1+\alpha)(1-d) - (1-\alpha)(1+d)]}{1-d^2}$$
$$= \frac{2q(\alpha - d)}{1-d^2} = \frac{4\alpha q(1-q)}{1-\alpha^2(2q-1)^2}.$$

- **The general bid–ask spread:** Multiplying the difference in posteriors by ΔV:

$$\boxed{Ask - Bid = \Delta V(q_B - q_S) = \Delta V \frac{4\alpha q(1-q)}{1-\alpha^2(2q-1)^2}.}$$

  $\hookrightarrow$ At $q = 1/2$, we recover $Ask - Bid = \alpha\Delta V$.
  $\hookrightarrow$ For fixed $0 < \alpha < 1$, the spread is largest at $q = 1/2$ and approaches zero as $q \to 0$ or $q \to 1$.

$\trianglerighteq$ **Economic interpretation:** The spread depends on informed trading (α) and the dealer's uncertainty about V (q).

In-Class Exercise 1: Learning from Trades

- Consider the Glosten–Milgrom economy with: $V_H = 120$, $V_L = 80$, $q = \Pr(V_H) = \frac{3}{5}$, $\alpha = \frac{1}{2}$.
- **Questions (20 minutes):**
  1. **Order probabilities and beliefs:** Construct the likelihood table $\Pr(B \mid V)$ and $\Pr(S \mid V)$. Compute $\Pr(B)$ and $\Pr(S)$, then use Bayes' rule to derive $q_B = \Pr(V_H \mid B)$ and $q_S = \Pr(V_H \mid S)$.
  2. **Competitive quotes:** Using the dealer's conditional zero-profit conditions, compute the ask, bid, and bid–ask spread. Compare both quotes with $\mathbb{E}[V]$. Why do they differ?
  3. **Learning from sequential trades:** Suppose the first customer buys. The dealer updates beliefs, and a new trader arrives without any additional public news.
     - Compute the **new bid** at which the dealer accepts a sell.
     - After observing a buy followed by a sell, what is the posterior probability of V_H?
  4. **More informed trading:** Back to the original prior $q = 3/5$ and increase α to $3/4$. Recompute the quotes and spread. Explain what changes and whether the distribution of fundamental values has changed.

Exercise 1 Solution (1/2): Posteriors and Quotes

- **Probabilities:** With $\alpha = 1/2$, an informed trader buys only when $V = V_H$, while an uninformed trader buys with probability $1/2$:

	V_H	V_L
$\Pr(B \mid V)$	$3/4$	$1/4$
$\Pr(S \mid V)$	$1/4$	$3/4$

  By the law of total probability: $\Pr(B) = \frac{3}{5}\frac{3}{4} + \frac{2}{5}\frac{1}{4} = \boxed{\frac{11}{20}}$, $\Pr(S) = \frac{3}{5}\frac{1}{4} + \frac{2}{5}\frac{3}{4} = \boxed{\frac{9}{20}}$.

- **Update beliefs:** Using Bayes' rule, conditional on the observed order:

$$q_B = \Pr(V_H \mid B) = \frac{\Pr(B \mid V_H)\Pr(V_H)}{\Pr(B)} = \frac{9/20}{11/20} = \boxed{\frac{9}{11}}, \quad q_S = \Pr(V_H \mid S) = \boxed{\frac{1}{3}}.$$

  ↪ A buy raises the probability of V_H from $3/5$ to $9/11$; a sell lowers it to $1/3$.

- **Competitive quotes:** Zero expected profit conditional on each trade direction implies:

$$Ask = \mathbb{E}[V \mid B] = 80 + 40\frac{9}{11} = \boxed{112.727}, \quad Bid = \mathbb{E}[V \mid S] = 80 + 40\frac{1}{3} = \boxed{93.333}.$$

  Thus:

$$Ask - Bid = \frac{640}{33} = \boxed{19.394}, \qquad \mathbb{E}[V] = 80 + 40\frac{3}{5} = 104.$$

↪ **Why do quotes differ from $\mathbb{E}[V]$?** A buy (sell) signals a higher (lower) fundamental value. Quoting 104 on both sides would expose the dealer to expected losses conditional on each order direction.

Exercise 1 Solution (2/2): Learning and the Spread

- **Step 4 — Sequential learning:** After the first buy, the dealer updates the prior to $\tilde{q} = 9/11$. With no new public information, a subsequent sell implies: $q_{BS} = \Pr(V_H \mid B, S) = (\frac{9}{11}\frac{1}{4})/(\frac{9}{11}\frac{1}{4} + \frac{2}{11}\frac{3}{4}) = \boxed{3/5}$ so that: $Bid^{\text{seq}} = 80 + 40q_{BS} = \boxed{104}$.
  ↪ The buy initially raises the probability of V_H, but the subsequent sell reverses that update.
  Opposite orders cancel their likelihood ratios, restoring the original prior $q = 3/5$.
- **Step 5 — More informed trading:** Reset to $q = 3/5$ and increase α from $1/2$ to $3/4$. Now $\Pr(B \mid V_H) = 7/8$ and $\Pr(B \mid V_L) = 1/8$. Thus:

$$\Pr(B) = \frac{23}{40}, \qquad \Pr(S) = \frac{17}{40}, \quad q_B = \frac{(3/5)(7/8)}{23/40} = \frac{21}{23}, \quad q_S = \frac{(3/5)(1/8)}{17/40} = \frac{3}{17}.$$

  The competitive quotes and spread become:

$$Ask = 80 + 40\frac{21}{23} = \boxed{116.522}, \quad Bid = 80 + 40\frac{3}{17} = \boxed{87.059}, \qquad Ask - Bid = \boxed{29.463}.$$

  ↪ The spread increases from 19.394 to 29.463: orders are more informative, while the fundamental values and their prior probabilities remain unchanged.
- **Step 6 — Diagnose the mistake:** α is the probability of an *informed trader*, not the probability of a buy. Dealer break-even requires: $\mathbb{E}[Ask - V \mid B] = 0$, $\mathbb{E}[V - Bid \mid S] = 0$, which also imply: $\boxed{\Pr(B)Ask + \Pr(S)Bid = \mathbb{E}[V].}$

▷ **The central insight:** Prices respond to information revealed by orders, not necessarily to changes in fundamental value.

Kyle (1/3): The Insider's Trading Decision

- *What if an informed trader can choose how much to trade, knowing that larger orders affect prices?*
- Consider a single auction with: $V \sim N(\mu, \sigma_V^2), \quad u \sim N(0, \sigma_u^2), \quad V \perp u.$
  - **Insider:** Observes fundamental V and chooses order x (+ive for buying, -ive for selling), w/out observing noise demand u.
  - **Dealers:** Observe only total order flow $y = x + u$ and set the competitive execution price $P(y) = \mathbb{E}[V \mid y]$.
- Linear pricing rule: $P(y) = \mu + \lambda y$,
  $\hookrightarrow$ a larger buy (sell) raises (lowers) the execution price. $\lambda > 0$ is the price impact of one share more.
- **Insider's problem:** Taking λ as given, the risk-neutral insider maximizes expected trading profits:

$$\max_x \ \mathbb{E}[(V - P(x+u))x \mid V] = \max_x \ \mathbb{E}[(V - \mu - \lambda(x+u))x \mid V]$$

$$= \max_x \ \underbrace{\{(V-\mu)x}_{\text{gain from private info}} - \underbrace{\lambda x^2\}}_{\text{cost of price impact}} \qquad \text{since } \mathbb{E}[u \mid V] = 0.$$

- Optimal trading (FOC): $V - \mu - 2\lambda x = 0$, with $\partial^2 \mathbb{E}[\pi \mid V]/\partial x^2 < 0 \implies \boxed{x = (V-\mu)/2\lambda}$.

$\hookrightarrow$ **Economic interpretation:** More valuable private information ($|V - \mu| \uparrow$) encourages larger trades, while greater price impact ($\lambda \uparrow$) discourages them. An additional share worsens the execution price on the *entire order*.

$\trianglerighteq$ The insider's FOC determines optimal trading *given* λ.

Kyle (2/3): Equilibrium

- We established that the insider optimally trades $x = \beta(V - \mu)$, with $\beta = 1/(2\lambda)$.
  How do dealers determine λ from the information contained in total order flow?
- Dealers observe $y = x + u = \beta(V - \mu) + u$, but cannot distinguish.
  Since V and u are independent Gaussian variables, (V, y) is jointly normal: $\mathbb{E}[y] = 0$, $\mathrm{Cov}(V, y) = \beta\sigma_V^2$, $\mathrm{Var}(y) = \beta^2\sigma_V^2 + \sigma_u^2$. Thus, the competitive pricing rule is:

$$P(y) = \mathbb{E}[V \mid y] = \mu + \frac{\mathrm{Cov}(V, y)}{\mathrm{Var}(y)} y = \mu + \underbrace{(\beta\sigma_V^2)/(\beta^2\sigma_V^2 + \sigma_u^2)}_{\lambda} y.$$

  $\hookrightarrow$ The price response to an order depends on how much information that order reveals about V.
- **Equilibrium consistency:** The dealer's pricing rule must agree with the insider's optimal trading strategy:

$$\underbrace{\beta\lambda = 1/2}_{\text{insider FOC}} \Longrightarrow 2\beta^2\sigma_V^2 = \beta^2\sigma_V^2 + \sigma_u^2 \Longrightarrow \boxed{\beta = \frac{\sigma_u}{\sigma_V}, \quad \lambda = \frac{\sigma_V}{2\sigma_u}},$$

$\hookrightarrow$ More noise trading ($\sigma_u \uparrow$) makes informed orders harder to identify, reducing price impact ($\lambda \downarrow$). The insider responds by trading more aggressively ($\beta \uparrow$).

! Dealer zero profit requires $P(y) = \mathbb{E}[V \mid y]$, **conditional on order flow**.

$\trianglerighteq$ **Kyle equilibrium:** Optimal insider trading and Bayesian dealer pricing determine each other simultaneously.

Kyle (3/3): Market Depth

- Kyle equilibrium: $x = \beta(V - \mu), \quad \beta = \frac{\sigma_u}{\sigma_V}, \quad P(y) = \mu + \lambda y, \quad \lambda = \frac{\sigma_V}{2\sigma_u}$.
  Does a more liquid market necessarily produce more informative prices?
- **Market depth:** $\lambda = \partial P / \partial y$ measures price impact; $1/\lambda$ measures mkt depth.
  - $\sigma_u \uparrow \implies \lambda \downarrow$: More noise trading makes informed orders harder to detect, reducing price impact.
  - ! The insider responds by trading more aggressively ($\beta \uparrow$).
- Equilibrium order flow: Since $x = \beta(V - \mu)$ and $y = x + u$, with $V \perp u$:

$$\mathrm{Var}(x) = \beta^2 \sigma_V^2 = \sigma_u^2, \qquad \mathrm{Var}(y) = \mathrm{Var}(x) + \mathrm{Var}(u) = \boxed{2\sigma_u^2}.$$

  $\hookrightarrow$ Insider trading increases proportionally with noise trading: their order variances are equal in equilibrium.
- **Price informativeness**: How much uncertainty about V remains after observing y? By joint normality:

$$\mathrm{Var}(V \mid y) = \sigma_V^2 - \frac{\mathrm{Cov}(V, y)^2}{\mathrm{Var}(y)} = \sigma_V^2 - \frac{\beta^2 \sigma_V^4}{2\sigma_u^2} = \boxed{\frac{\sigma_V^2}{2}}.$$

  $\hookrightarrow$ Prices reveal half of the prior variance, regardless of σ_u.
- Insider's expected profit: Using the FOC $\lambda\beta = 1/2$: $\mathbb{E}[(V - P(y))x] = \beta(1 - \lambda\beta)\sigma_V^2 = \boxed{\sigma_u \sigma_V / 2}$.
  $\hookrightarrow$ noisier trading allows the insider to exploit private information more profitably.

▷ **Depth $\neq$ informativeness:** Noisier trading makes prices less sensitive to orders, but induces larger informed trades. In Kyle equilibrium, these effects offset in the information revealed by prices.

Inventory Risk, Spreads, and Market Depth

- In Glosten-Milgrom/Kyle, dealers are risk-neutral and face no inventory costs. *What if dealers must hold risky positions?*
- **Inventory risk:** Dealers absorb imbalances between buyers and sellers.

  A dealer holding a large inventory faces losses if the asset price falls before the position can be unwound.

  ↪ A risk-averse dealer with a **long** inventory may lower both quotes: a lower ask encourages customers to buy from the dealer, while a lower bid discourages additional customer sales. Conversely, a **short** inventory encourages higher quotes.
- **Two distinct mechanisms:**
  - **Adverse selection:** Orders reveal information about fundamental value.
  - **Inventory risk:** Dealers require compensation for carrying risky positions; quotes adjust to encourage trades that reduce unwanted exposure.

Dimension of market liquidity	Economic interpretation
Tightness (spread)	Cost of immediately buying and reselling a small position.
Depth (price impact)	How much the execution price changes with the size of an order.
Resiliency	How quickly prices and available liquidity recover after an order imbalance.

↪ A narrow bid-ask spread does not guarantee low trading costs: large orders may still move prices substantially in shallow mkt.

▷ **Liquidity is multidimensional:** Trading costs depend on both the price quotedand the mkt's ability to absorb/recover.

Trading Costs: Amihud–Mendelson (1/2)

- *How does the cost of trading (rather than information) affect the value investors assign to an asset?*
- **Setup:** Consider an asset paying a constant dividend flow $d > 0$ per year, with continuous discount rate $\rho > 0$.
  - The investor buys at the stationary ask P and sells at the bid $(1 - s)P$, with $s \in [0, 1)$.
  - Liquidation occurs at $T \sim \text{Exp}(h)$, with expected holding period $\mathbb{E}[T] = 1/h$.
- **Value of holding the asset:** Dividends are received until liquidation at T, followed by resale at the bid:

$$\mathbb{E}\Big[\underbrace{\int_0^T e^{-\rho t} d\,\mathrm{d}t}_{\text{dividends until liquidation}} + \underbrace{e^{-\rho T}(1-s)P}_{\text{resale proceeds at } T}\Big].$$

  Since $T \sim \text{Exp}(h)$, recall: $\Pr(T > t) = e^{-ht}$, $\quad f_T(t) = he^{-ht}$. For $a > 0$, $\int_0^\infty e^{-at}\mathrm{d}t = \left[-e^{-at}/a\right]_0^\infty = 1/a$, since $e^{-at} \to 0$ as $t \to \infty$.
- **Expected discounted dividends:** A dividend at t is received only if the investor still holds the asset:

$$\mathbb{E}\left[\int_0^T e^{-\rho t} d\,\mathrm{d}t\right] = \int_0^\infty \underbrace{e^{-\rho t} d}_{\text{discounted flow}} \underbrace{\Pr(T > t)}_{\text{still holding}}\,\mathrm{d}t = d\int_0^\infty e^{-(\rho+h)t}\mathrm{d}t = \boxed{\frac{d}{\rho + h}}.$$

- **Expected discounted resale:** The investor sells at the random date T. Averaging over possible liquidation dates:

$$\mathbb{E}[e^{-\rho T}(1-s)P] = (1-s)P\int_0^\infty \underbrace{e^{-\rho t}}_{\text{discounting}} \underbrace{he^{-ht}}_{\text{liquidation density}}\,\mathrm{d}t = \boxed{\frac{h(1-s)P}{\rho + h}}.$$

Trading Costs: Amihud–Mendelson (2/2)

- The expected discounted proceeds from holding the asset until liquidation are: $(d + h(1-s)P)/(\rho + h)$.
  ↪ *How does the bid–ask spread affect the asset's purchase price and required dividend yield?*
- In a stationary (one-clientele) reservation-value benchmark, the purchase price equals the exp. discounted proceeds:

$$P = \frac{d + h(1-s)P}{\rho + h} \Longrightarrow P(\rho + h) = d + hP - hsP \Longrightarrow \boxed{P = \frac{d}{\rho + hs}}$$

- **The dividend yield:** Rearranging the price equation: $\frac{d}{P} = \underbrace{\rho}_{\text{discount rate}} + \underbrace{hs}_{\text{annual. trading-cost burden}}$
  - ρ: the continuous discount rate applied to future cash flows.
  - h: the liquidation intensity (expected holding period $1/h$); recall s: the proportional loss upon resale.

  ↪ The pre-cost dividend yield must compensate for both discounting and expected trading costs.
- **Comparative statics:** Holding d and ρ fixed:

$$\frac{\partial P}{\partial s} = -\frac{dh}{(\rho + hs)^2} < 0, \qquad \frac{\partial P}{\partial h} = -\frac{ds}{(\rho + hs)^2} < 0 \quad (s > 0).$$

  ↪ $s \uparrow$: A larger spread reduces resale proceeds, lowering the price investors are willing to pay.
  ↪ $h \uparrow$: More frequent exp. liquidation increases the cost burden; shorter-horizon investors value the same illiquid asset less.

▷ The same asset can be more valuable to long-horizon investors, who expect to incur the resale cost less frequently.

Holding-Period Clienteles and Liquidity Premia

- *If investors have different holding horizons, who will hold the less liquid assets?*
- Consider investor types j, with liquidation intensities $h_j > 0$, and assets i with dividend yields $y_i = d_i/P_i$ and spreads s_i.
  - Type j selects assets offering the highest yield net of expected trading costs: $r_j^* = \max_i\{y_i - h_j s_i\}$.
  - A larger h_j means a shorter expected holding period and greater sensitivity to the spread. Investors cannot short.
- **Which investor determines the asset's yield?** For type j, holding asset i is optimal only if its net yield equals the best available alternative: $y_i - h_j s_i = r_j^* \iff y_i = r_j^* + h_j s_i$.
  By definition of r_j^*, every asset must satisfy: $\forall j,\ y_i \le r_j^* + h_j s_i$.
  Since the asset has positive supply, some investor must hold it and attain equality: $y(s_i) = \min_j\{r_j^* + h_j s_i\}$.

↪ At a given spread, the investor requiring the **lowest dividend yield** is willing to hold the asset. Other investors prefer their alternatives. The market yield adjusts to this investor's valuation.

- **Why do investors sort by holding horizon?** Suppose type j holds asset a and type k holds asset b, with $h_j > h_k$. Their optimal choices imply:

$$y_a - h_j s_a \ge y_b - h_j s_b, \quad y_b - h_k s_b \ge y_a - h_k s_a \qquad \Longrightarrow \qquad \boxed{(h_j - h_k)(s_b - s_a) \ge 0}.$$

  ↪ Since $h_j > h_k$, necessarily $s_b \ge s_a$: the longer-horizon investor holds the asset with the higher spread.

≥ **Liquidity premia and holding horizons:** Higher spreads require higher dividend yields. However, as spreads increase, assets are held by investors with longer horizons ($h_j \downarrow$), who require a smaller yield increase for additional trading costs.
↪ the relationship between spreads and yields becomes flatter.

Liquidity Risk: Acharya–Pedersen

- Previously, investors require compensation for **expected** trading costs.
  What if liquidity itself is stochastic, so selling an asset becomes more costly precisely in bad states?

- An investor at t buys one share at $P_{i,t}$, receives dividend $D_{i,t+1}$, and must sell at $t+1$.
  - The sale incurs a random cost $C_{i,t+1}$ per share: the investor receives $P_{i,t+1} - C_{i,t+1}$, rather than $P_{i,t+1}$.
  - ↪ The total payoff net is: $P_{i,t+1} + D_{i,t+1} - C_{i,t+1}$.

- **Gross vs. net returns:** Dividing each component by the purchase price:

$$R^g_{i,t+1} = \underbrace{(P_{i,t+1} + D_{i,t+1})/P_{i,t}}_{\text{gross return before costs}}, \qquad c_{i,t+1} = \underbrace{(C_{i,t+1})/P_{i,t}}_{\text{liqu. cost per dollar}}, \qquad R^{g,\text{net}}_{i,t+1} = R^g_{i,t+1} - c_{i,t+1}.$$

  Writing simple returns as $r_i = R^g_i - 1$, we obtain: $r^{\text{net}}_{i,t+1} = r_{i,t+1} - c_{i,t+1}$

- **Equilibrium environment:** The Acharya–Pedersen model assumes:
  - Two-period-lived investors with CARA preferences and conditionally normal net payoffs.
  - Borrowing/lending at the safe rate, but no short sales of stocks.
    ↪ In equilibrium, investors hold positive positions in the risky market portfolio.

- **Two different sources of liquidity compensation:**
  - **Expected cost:** $\mathbb{E}_t[c_{i,t+1}]$ reduces the return investors expect to receive, even if the cost is perfectly predictable.
  - **Liquidity risk:** Unexpected changes in $c_{i,t+1}$ matter when they coincide with adverse market conditions, making the asset less valuable precisely when investor wealth is low.

Liquidity Risk: From Portfolios to Mkt Pricing

- *If investors care about returns net of liquidation costs, how does portfolio choice determine equilibrium risk premia?*
- **Investor's problem:** Suppress index t. Define net simple returns $r^n = r - c$, expected excess returns $e = \mathbb{E}[r^n] - r_f \mathbf{1}$, and their covariance matrix $\Sigma = \mathrm{Var}(r^n)$ (nonsingular).

  Investor j, with absolute risk aversion $a_j > 0$, chooses dollar positions $x^j \in \mathbb{R}^N$. Under CARA preferences and normally distributed net returns:

$$\max_{x^j} \quad (x^j)'e - \frac{a_j}{2}(x^j)'\Sigma x^j, \qquad \text{FOC: } e - a_j \Sigma x^j = 0 \quad \Longrightarrow \quad \boxed{x^j = (\Sigma^{-1}e)/a_j}.$$

  $\hookrightarrow$ Higher expected net excess returns encourage investment; greater covariance risk discourages it.
  Stock short sales are prohibited, but positive equilibrium holdings make the FOCs interior.
- **Market clearing:** Let $m = \sum_j x^j$ denote aggregate risky dollar holdings (the capitalization vector), with total value $V_M = \mathbf{1}'m > 0$ and market weights $w_M = m/V_M$. Aggregate risk aversion $a = (\sum_j 1/a_j)^{-1}$:

$$m = \sum_j \frac{1}{a_j}\Sigma^{-1}e = \frac{1}{a}\Sigma^{-1}e \quad \Longrightarrow \quad \boxed{e = aV_M \Sigma w_M}.$$

  $\hookrightarrow$ Market clearing replaces individual holdings with the market portfolio.
- **Which covariance is priced?** Define the net market return $r_M^n = w_M' r^n$. Since Σ is the covariance matrix of net returns:

$$\underbrace{e_i}_{\mathbb{E}[r_i^n]-r_f} = aV_M \underbrace{(\Sigma w_M)_i}_{\mathrm{Cov}(r_i^n, r_M^n)}, \qquad \underbrace{e_M}_{w_M' e = \mathbb{E}[r_M^n]-r_f} = aV_M \underbrace{w_M'\Sigma w_M}_{\mathrm{Var}(r_M^n)}.$$

- **Equilibrium pricing:** Both the individual asset and the market earn premia proportional to their exposure to **net market-return risk**, with the same coefficient aV_M.

Liquidity Risk: Four Covariance Terms

- **Net-return CAPM:** divide the asset's excess-return equation by the market's to eliminate aV_M. Restoring conditioning at t:

$$\mathbb{E}_t[r_i - c_i] - r_f = \underbrace{\big(\mathbb{E}_t[r_M - c_M] - r_f\big)}_{\mathrm{RP}^n_{M,t}} \frac{\mathrm{Cov}_t(r_i - c_i, r_M - c_M)}{\mathrm{Var}_t(r_M - c_M)}.$$

  $\hookrightarrow$ The relevant risk exposure is the covariance between the returns *actually received* by an investor and by the market, **both net of liquidation costs**.

- Expanding the cov: $\mathrm{Cov}_t(r_i - c_i, r_M - c_M) = \mathrm{Cov}_t(r_i, r_M) + \mathrm{Cov}_t(c_i, c_M) - \mathrm{Cov}_t(r_i, c_M) - \mathrm{Cov}_t(c_i, r_M)$. Since $\mathbb{E}_t[r_i - c_i] = \mathbb{E}_t[r_i] - \mathbb{E}_t[c_i]$, defining $D_t = \mathrm{Var}_t(r_M - c_M) > 0$ gives:

$$\mathbb{E}_t[r_i] - r_f = \mathbb{E}_t[c_i] + \frac{\mathrm{RP}^n_{M,t}}{D_t}\Big[\mathrm{Cov}_t(r_i, r_M) + \mathrm{Cov}_t(c_i, c_M) - \mathrm{Cov}_t(r_i, c_M) - \mathrm{Cov}_t(c_i, r_M)\Big].$$

- **Economic interpretation of the four covariances:**
  - $+\mathrm{Cov}(r_i, r_M)$ — **Market risk:** A low asset return when the market falls is costly to investors.
  - $+\mathrm{Cov}(c_i, c_M)$ — **Common liquidity risk:** The asset becomes expensive to sell when selling is expensive.
  - $-\mathrm{Cov}(r_i, c_M)$ — **Return sensitivity to liquidity:** An asset paying more when market trading costs rise provides insurance and requires a lower return.
  - $-\mathrm{Cov}(c_i, r_M)$ — **Liquidity sensitivity to returns:** If selling costs rise when market returns fall, the covariance is negative and the required return increases.

$\trianglerighteq$ **Expected illiquidity versus liquidity risk:** $\mathbb{E}_t[c_i]$ compensates for the expected cost of selling; the covariance terms compensate for how returns and selling costs behave in aggregate market conditions.

TBD