

Asset Pricing Theory: S04

Production-Based Asset Pricing

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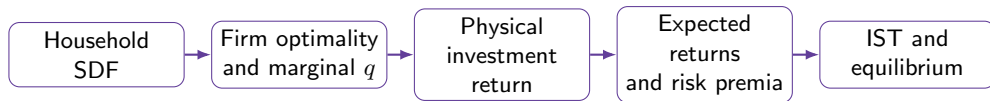
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Structure of the course

- Email: amedeo.andriollo@dauphine.psl.eu
 - Grading follows a 30-70 rule: 70% final exam, 30% project/homework (Nov. 6th).
 - Office hours: feel “free” to (DO) come: B612.
 - Slides/materials are updated regularly. Report a typo/mistake: corrections help the class.
 - This session:
 - Firm optimality, marginal q , investment returns, expected returns, investment-specific technology, and equilibrium..
 - Main references: Campbell (2018, Ch.7); Cochrane (1991); Koijen and Van Nieuwerburgh (2020); Cochrane (2005, Ch.2.2 and 21.3).
- ▷ Market frictions.

Today's Route: From Investors to Firms

- **Sessions 1-2:** NA gives us a pricing restriction, while household/investor links the SDF to marginal utility. **Session 3:** Investors choose consumption and portfolios dynamically, given the investment opportunity set.
- **This session:** *What returns does physical investment generate, and how are those returns priced?*



- Household: marginal rate of substitution determines how future payoffs are valued across dates/states.
- **Firms:** marginal rate of transformation determines what future payoffs can be generated by investing resources today.
- **The connection:** Firm optimality gives us a return on physical investment that must satisfy the same SDF pricing restriction as financial assets.

What Can Firm Optimality Tell Us?

- A firm can i) distribute resources to its shareholders today or ii) invest them in productive capital to generate future payoffs.
↪ **Trade-off:** investment reduces current distributions but increases future cash flows and value.
- At the optimum, (marginal) cost of investing must equal the present value of its (marginal) benefit:

$$\underbrace{\text{Marginal cost at } t}_{\text{foregone current payout}} = \mathbb{E}_t \left[M_{t+1} \underbrace{\text{Marginal payoff at } t+1}_{\text{cash flow} + \text{continuation value}} \right].$$

- Dividing by the current marginal cost gives us the familiar:

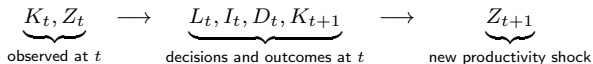
$$1 = \mathbb{E}_t \left[M_{t+1} R_{t+1}^{I,g} \right],$$

where $R_{t+1}^{I,g}$ is the gross return on physical investment.

- ↪ **The key insight:** Firm optimality allows us to construct an economically meaningful return that must be priced by the same SDF as financial assets.
- ▷ This provides a production-based pricing restriction.

Back to Discrete Time

- Discrete time, $t = 0, 1, 2, \dots$. The consumption good is our numeraire: all prices/returns are in real terms.
 - **Agent decisions:** At date t , firms inherit the capital stock K_t and observe productivity Z_t . Given the available information $\mathcal{F}_t$:
 - Firms choose labor L_t and investment I_t , thus determining current dividends D_t .
 - (Households choose consumption and their portfolio holdings.)
- Production at t uses the capital stock K_t . Investment undertaken today determines K_{t+1} , which becomes productive the next period.



- As in the previous Sessions, we distinguish gross and net returns:

$$R_{i,t+1}^g = \frac{P_{i,t+1} + D_{i,t+1}}{P_{i,t}}, \quad R_{i,t+1}^{net} = R_{i,t+1}^g - 1,$$

where $P_{i,t}$ is the ex-dividend price and $D_{i,t+1}$ is the next-period payout.

→ We will denote the gross return on physical investment by $R_{t+1}^{I,g}$.

Production, Investment, and Resources

- A competitive firm combines physical capital K_t and effective labor $Z_t L_t$ (Z_t denotes labor productivity) to produce consumption goods: $Y_t = F(K_t, Z_t L_t)$. The concave production function has positive marginal products: $F_K > 0$, $F_L > 0$, $F_{KK} < 0$, $F_{LL} < 0$. Real wage is w_t .
- Investment increases the capital next period using resources that could be distributed to shareholders:

$$K_{t+1} = (1 - \delta)K_t + I_t, \quad D_t = Y_t - w_t L_t - I_t - G(I_t, K_t),$$

where $\delta \in (0, 1)$ is the depreciation rate and $G(I_t, K_t)$ captures the adjusting investment costs.

↪ we consider a quadratic specification:

$$G(I, K) = \frac{\kappa}{2} \left(\frac{I}{K} - \delta \right)^2 K, \quad \kappa > 0, \quad K > 0.$$

- Adjustment costs are zero when investment exactly replaces depreciated capital, $I/K = \delta$.
- Deviations from this replacement rate are costly. A larger κ implies greater adjustment costs.

- ! Investment uses resources today to create capital tomorrow VS. adjustment costs reduce current dividends.
- ▷ We initially assume interior investment choices with positive marginal investment costs.

The Household Side of the Economy

- The household derives utility from consumption C_t and leisure $\ell_t = 1 - L_t^s$ (L_t^s is labor supply:

$$\max_{\{C_t, L_t^s, \theta_{i,t}\}} \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t u(C_t, \ell_t), \quad 0 < \beta < 1, \quad u_C, u_\ell > 0.$$

- At each date, the household allocates its resources between consumption and financial assets:

$$\underbrace{C_t + \sum_i P_{i,t} \theta_{i,t}}_{\text{consumption and portfolio purchases}} = \underbrace{w_t L_t^s + \sum_i (P_{i,t} + D_{i,t}) \theta_{i,t-1}}_{\text{labor income and existing portfolio value}},$$

where $P_{i,t}$ is the ex-dividend price, $D_{i,t}$ the dividend, and $\theta_{i,t}$ the holdings chosen at t .

- Let B_t denote the difference between the RHS and LHS of the budget constraint. The Lagrangian is:

$$\mathcal{L}^H = \mathbb{E}_0 \sum_{t=0}^{\infty} \beta^t [u(C_t, 1 - L_t^s) + \eta_t B_t],$$

- At an interior optimum, the first-order conditions imply:

$$\eta_t = u_C(C_t, \ell_t), \quad \boxed{\frac{u_\ell(C_t, \ell_t)}{u_C(C_t, \ell_t)} = w_t.}$$

Household Optimality and the SDF

- The household decides how many shares $\theta_{i,t}$ of asset i to hold:
 - Purchasing one additional share costs $P_{i,t}$ units of consumption today.
 - Next period, the share delivers $P_{i,t+1} + D_{i,t+1}$, whose value is uncertain at t .
- FOC wrt $\theta_{i,t}$, and by virtue of law of iterated expectations:

$$\underbrace{\eta_t P_{i,t}}_{\text{marginal utility cost today}} = \beta \mathbb{E}_t \left[\underbrace{\eta_{t+1} (P_{i,t+1} + D_{i,t+1})}_{\text{marginal utility benefit tomorrow}} \right].$$

- Recall that $\eta_t = u_C(C_t, \ell_t)$. Dividing the first-order condition by $\eta_t P_{i,t}$ gives:

$$1 = \mathbb{E}_t \left[M_{t+1} R_{i,t+1}^g \right], \quad M_{t+1} = \beta \frac{u_C(C_{t+1}, \ell_{t+1})}{u_C(C_t, \ell_t)}.$$

→ **Why conditional expectations?** The household chooses its portfolio using the information available at date t .

What Does the Firm Maximize?

- A firm acts in the interest of its shareholders by choosing labor and investment to maximize the present value of its future dividends, taking the household's SDF M_{t+1} as given.
- Define: $\Lambda_0 = 1$, $\Lambda_t = \prod_{s=1}^t M_s$, $M_{t,t+j} = \Lambda_{t+j}/\Lambda_t$, $M_{t,t} = 1$, where $M_{t,t+j}$ discounts a payoff at $t+j$ back to date t . Let V_t denote the post-dividend value of the firm, including the dividend distributed at date t :

$$V_t(K_t) = \max_{\{I_{t+j}, L_{t+j}\}_{j \geq 0}} \mathbb{E}_t \left[\sum_{j=0}^{\infty} M_{t,t+j} D_{t+j} \right] = \max_{I_t, L_t} \{D_t + \mathbb{E}_t [M_{t+1} V_{t+1}(K_{t+1})]\}.$$

- Post-dividend value: firm's mkt price after the current dividend has been distributed is

$$P_t = V_t - D_t = \mathbb{E}_t [M_{t+1} V_{t+1}].$$

→ The firm chooses investment by comparing the resources distributed to shareholders today with the SDF-discounted value of the additional future dividends that investment generates.

- subject to the constraints:

$$D_t = F(K_t, Z_t L_t) - w_t L_t - I_t - G(I_t, K_t),$$

$$K_{t+1} = (1 - \delta)K_t + I_t.$$

The Firm's Maximization Problem

- The Lagrangian introduces a multiplier q_t on the capital accumulation constraint:

$$\mathcal{L}^F = \mathbb{E}_0 \sum_{t=0}^{\infty} \Lambda_t [D_t + q_t((1 - \delta)K_t + I_t - K_{t+1})].$$

→ Economic interpretation of q_t :

- Each period has its own capital accumulation constraint, so q_t is generally time-varying and state-dependent.
 - It is the **shadow value of installed capital**: the value at t of an additional unit of productive capital available at $t + 1$.
 - Multiplying q_t by Λ_t expresses this value in date-0 valuation units: q_t is measured in date- t consumption units.
- Differentiating $\mathcal{L}^F$ wrt I_t :

$$\frac{\partial \mathcal{L}^F}{\partial I_t} = 0 \implies -1 - G_I(I_t, K_t) + q_t = 0 \implies \boxed{q_t = 1 + G_I(I_t, K_t)}.$$

Because $G_I(I_t, K_t) = \kappa \left(\frac{I_t}{K_t} - \delta \right)$, then $\boxed{I_t/K_t = \delta + (q_t - 1)/\kappa}$.

Higher marginal q encourages investment; larger adjustment costs (higher κ) reduce its responsiveness.

- Differentiating $\mathcal{L}^F$ wrt L_t :

$$\frac{\partial \mathcal{L}^F}{\partial L_t} = 0 \implies \boxed{Z_t F_2(K_t, Z_t L_t) = w_t}.$$

The firm hires labor until its marginal product, measured in consumption goods, equals the real wage.

Firm Optimality: The Capital Euler Equation

- Differentiating the Lagrangian wrt K_{t+1} requires considering two consecutive periods:
i) at t , an additional unit of K_{t+1} has a marginal cost q_t , ii) at $t + 1$, the additional capital affects current dividends and the value of the remaining capital stock.
- The envelope condition:** the marginal value of capital inherited at $t + 1$ is:

$$V_{K,t+1} = \frac{\partial D_{t+1}}{\partial K_{t+1}} + (1 - \delta)q_{t+1} = \underbrace{F_{K,t+1} - G_{K,t+1}}_{\text{marginal contribution to dividends}} + \underbrace{(1 - \delta)q_{t+1}}_{\text{continuation value}}.$$

By the envelope theorem, the derivative is evaluated at the optimal choices of labor and investment, w/out differentiating those choices wrt capital.

- The capital FOC:** Differentiating the two relevant terms of $\mathcal{L}^F$ wrt K_{t+1} gives:

$$\underbrace{-\Lambda_t q_t}_{\text{date } t} + \underbrace{\Lambda_{t+1} [F_{K,t+1} - G_{K,t+1} + (1 - \delta)q_{t+1}]}_{\text{date } t+1}.$$

Since K_{t+1} is chosen at t , its FOC must hold conditional on information available at that date:

$$0 = -\Lambda_t q_t + \mathbb{E}_t [\Lambda_{t+1} V_{K,t+1}] = \Lambda_t \{ -q_t + \mathbb{E}_t [M_{t+1} V_{K,t+1}] \},$$

where we used $\Lambda_{t+1} = \Lambda_t M_{t+1}$ and the fact that Λ_t is known at t .

- Dividing by $\Lambda_t > 0$ gives the **capital Euler equation**:

$$q_t = \mathbb{E}_t [M_{t+1} V_{K,t+1}] = \mathbb{E}_t [M_{t+1} (F_{K,t+1} - G_{K,t+1} + (1 - \delta)q_{t+1})].$$

▷ The marginal cost of installing capital today equals the SDF-discounted marginal value of capital tomorrow.

The Gross Return on Physical Investment

- Following Cochrane (1991), suppose the firm increases investment at t by $\varepsilon > 0$. This raises next-period capital by ε (i.e., $dK_{t+1} = \varepsilon$), while reducing current dividends: $dD_t = -(1 + G_{I,t})\varepsilon = -q_t\varepsilon$.
- To restore the original capital stock at $t + 2$, the firm reduces next-period investment by $(1 - \delta)\varepsilon$:

$$dI_{t+1} = -(1 - \delta)\varepsilon, \quad dK_{t+2} = (1 - \delta)\varepsilon + dI_{t+1} = 0.$$

$$\implies dD_{t+1} = (F_{K,t+1} - G_{K,t+1})\varepsilon - (1 + G_{I,t+1})dI_{t+1} = [F_{K,t+1} - G_{K,t+1} + (1 - \delta)q_{t+1}]\varepsilon.$$

- The ratio of the marginal payoff tomorrow to the marginal cost today defines

$$R_{t+1}^{I,g} = \frac{dD_{t+1}}{-dD_t} = \frac{\overbrace{F_{K,t+1} - G_{K,t+1}}^{\text{marginal cash flow}} + \overbrace{(1 - \delta)q_{t+1}}^{\text{continuation value}}}{\underbrace{q_t}_{\text{marginal cost at } t}}.$$

- **The pricing equation:** dividing the capital Euler equation by $q_t > 0$: $1 = \mathbb{E}_t \left[M_{t+1} R_{t+1}^{I,g} \right]$.

W/out adjustment costs, $q_t = q_{t+1} = 1$, so $R_{t+1}^{I,g} = F_{K,t+1} + 1 - \delta$.

- The investment return is an **intertemporal marginal rate of transformation**: it measures the additional resources available tomorrow from investing one unit of resources today.
- ⊇ This is the return on **marginal physical investment**, not yet the firm's stock return.

From Physical Investment to Financial Returns

- **What does the investment Euler equation tell us?** We established: $1 = \mathbb{E}_t \left[M_{t+1} R_{t+1}^{I,g} \right]$.
 - ↪ this pricing restriction does **not** imply that $M_{t+1} = 1/R_{t+1}^{I,g}$ in every state of the economy. Suppose there are two possible states at $t+1$, with probabilities π_1 and π_2 . The pricing equation becomes: $\pi_1 M_1 R_1^{I,g} + \pi_2 M_2 R_2^{I,g} = 1$, so one equation for two M 's unknown.
 - Investment allows firms to transfer across dates, but not between states.
 - ↪ identifying the SDF requires additional pricing restrictions or sufficiently rich production opportunities.
- *Can a financial portfolio replicate physical investment?* Suppose a traded portfolio delivers exactly the payoff $R_{t+1}^{I,g}$ in every state. By the SDF pricing restriction, its price:

$$p_t^{\text{mimic}} = \mathbb{E}_t \left[M_{t+1} R_{t+1}^{I,g} \right] = 1.$$

so that its gross return coincides with the physical investment return: $R_{t+1}^{\text{mimic},g} = \frac{R_{t+1}^{I,g}}{p_t^{\text{mimic}}} = R_{t+1}^{I,g}$.

Complete markets guarantee portfolio existence, although spanning this particular payoff is sufficient.

- ↪ Physical investment returns and traded portfolio returns can coincide state by state when their payoffs are identical. Satisfying the same pricing equation alone does not guarantee this equality.

In-Class Exercise 1

- Consider a competitive firm with an initial capital stock $K_t = 100$, a depreciation rate $\delta = 0.10$, and:

$$G(I, K) = 2 \left(\frac{I}{K} - 0.10 \right)^2 K, \quad \kappa = 4.$$

- Suppose that next-period marginal conditions are known at date t :

$$\frac{I_{t+1}}{K_{t+1}} = 0.20, \quad F_{K,t+1} = 0.18, \quad R_{f,t}^g = 1.25.$$

(No uncertainty between t and $t + 1$).

- Questions:**

- Marginal adjustment costs:** Derive $G_I(I, K)$ and $G_K(I, K)$.
- Marginal value of capital:** Using the firm's optimality conditions, compute q_{t+1} and the marginal payoff $V_{K,t+1}$.
- Optimal investment and returns:** Determine q_t and the optimal investment I_t . Compute the gross and net returns on physical investment, verifying that the capital Euler equation holds.
- An incorrect investment decision:** Suppose the manager ignores the adjustment-cost term $-G_{K,t+1}$ when evaluating the marginal payoff. How does this affect the estimated q_t and the resulting investment decision?

Exercise 1 Solution (1/2)

- **Marginal adjustment costs:** Recall that $G(I, K) = \frac{\kappa}{2} K(x - \delta)^2$.

If we set $x = \frac{I}{K}$, then: $\frac{\partial x}{\partial I} = \frac{1}{K}$ and $\frac{\partial x}{\partial K} = -\frac{I}{K^2} = -\frac{x}{K}$.

- **Derivative wrt investment:** Holding K fixed: $G_I(I, K) = \frac{\kappa}{2} K \cdot 2(x - \delta) \frac{1}{K} = \kappa(x - \delta)$.
- **Derivative wrt capital:** Holding investment I fixed, both K and the investment rate $x = I/K$ change:

$$G_K(I, K) = \frac{\kappa}{2}(x - \delta)^2 + \frac{\kappa}{2} K \cdot 2(x - \delta) \left(-\frac{x}{K}\right) = \frac{\kappa}{2}(\delta^2 - x^2).$$

- **Marginal value of capital at $t + 1$:** Using $x_{t+1} = 0.20$ and $\kappa = 4$, the investment FOC gives

$$q_{t+1} = 1 + 4(0.20 - 0.10) = 1.40, \quad G_{K,t+1} = 2(0.10^2 - 0.20^2) = -0.06.$$

Applying the envelope condition:

$$\begin{aligned} V_{K,t+1} &= F_{K,t+1} - G_{K,t+1} + (1 - \delta)q_{t+1} \\ &= \underbrace{0.18 + 0.06}_{\text{marginal cash flow} = 0.24} + \underbrace{0.90(1.40)}_{\text{continuation value} = 1.26} = \boxed{1.50}. \end{aligned}$$

- **Economic interpretation:** Since $G_{K,t+1} < 0$, additional capital reduces adjustment costs when investment is held fixed. This generates a marginal benefit of 0.06, in addition to the marginal product of capital.

Exercise 1 Solution (2/2)

- **The marginal value of investment at t :** With no uncertainty, the SDF is given by the inverse of the gross risk-free return:

$$M_{t+1} = \frac{1}{R_{f,t}^g} = \frac{1}{1.25} = 0.80.$$

Using the capital Euler equation and $V_{K,t+1} = 1.50$: $q_t = \mathbb{E}_t[M_{t+1}V_{K,t+1}] = 1.20$.

- **Optimal investment:** From the investment FOC, $q_t = 1 + G_I(I_t, K_t)$:

$$\frac{I_t}{K_t} = \delta + \frac{q_t - 1}{\kappa} = 0.10 + \frac{1.20 - 1}{4} = 0.15 \implies I_t = 100(0.15) = 15.$$

- **The return on physical investment:** The ratio of the marginal payoff tomorrow to the marginal cost today gives: $R_{t+1}^{I,g} = V_{K,t+1}/q_t = 1.50/1.20 = 1.25$. The net return is 25%, and the Euler equation is verified: $\mathbb{E}_t[M_{t+1}R_{t+1}^{I,g}] = 0.80(1.25) = 1$.
- **What if the manager ignores $-G_{K,t+1}$?** Omitting the adjustment-cost benefit understates the payoff:

$$\hat{V}_{K,t+1} = 0.18 + 0.90(1.40) = 1.44, \quad \hat{q}_t = 0.80(1.44) = 1.152, \quad \hat{I}_t = 100 \left(0.10 + \frac{1.152 - 1}{4} \right) = 13.8.$$

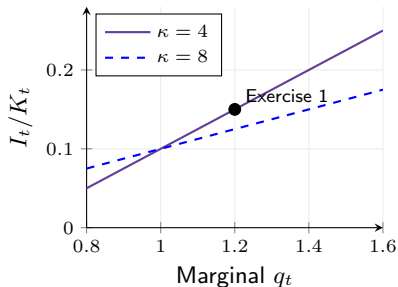
- ↪ **Economic interpretation:** The manager underinvests by $15 - 13.8 = 1.2$ units. At $\hat{I}_t = 13.8$, the true marginal value of investment is $q_t = 1.20$, whereas its marginal cost is only $1 + G_I = 1.152$.
↪ additional investment would increase firm value.

q-Theory: Value and Investment Move Together

- **The investment decision:** The firm's FOC equates the marginal value of installed capital to its marginal cost: $\frac{I_t}{K_t} = \delta + \frac{q_t - 1}{\kappa}$.
- **The investment response:** Holding technology fixed:
 $\frac{\partial(I_t/K_t)}{\partial q_t} = \frac{1}{\kappa} > 0$. Higher marginal q encourages investment, while larger adjustment costs (higher κ) reduce its responsiveness.
- **What determines q_t ?** Recall that

$$q_t = \mathbb{E}_t[M_{t+1} V_{K,t+1}].$$

Its value reflects both the expected marginal payoff of capital and the SDF used to discount that payoff. Higher q_t may therefore reflect improved future profitability or lower discount rates.



→ **What should we take home from the exercise?**

1. **Marginal valuation drives investment:** With $q_t = 1.20$ and $\kappa = 4$, the optimal investment rate is 15%. A larger κ would imply a smaller investment response to the same marginal valuation.
2. **The entire marginal payoff matters:** Ignoring the adjustment-cost benefit $-G_{K,t+1}$ understates q_t and reduces investment from 15 to 13.8. Investment decisions must account for both current marginal cash flows and the continuation value of capital.

Marginal q versus Average Q

- We have established that investment depends on marginal q_t , the shadow value of an additional unit of installed capital. As not directly observable, can we relate it to the firm's mkt value?
- Suppose production and adjustment costs are homogeneous of degree one, the firm is pricetaker, and no fixed costs or additional sources of rents.
 - Scaling capital, labor, and investment by the same factor $a > 0$ scales dividends and firm value proportionally:

$$V_t(aK_t) = aV_t(K_t).$$

- By Euler's theorem for homogeneous functions: $V_t(K_t) = V_{K,t}K_t \implies V_{K,t} = V_t/K_t$.
- **From marginal q to average Q :** Recall the capital Euler equation: $q_t = \mathbb{E}_t [M_{t+1}V_{K,t+1}]$. Substituting the homogeneity result gives:

$$q_t = \mathbb{E}_t \left[M_{t+1} \frac{V_{t+1}}{K_{t+1}} \right] = \frac{\mathbb{E}_t [M_{t+1}V_{t+1}]}{K_{t+1}} = \boxed{\frac{P_t}{K_{t+1}} \equiv Q_t},$$

where K_{t+1} is known at t and P_t is the firm's post-dividend market value.

- Under constant returns to scale, marginal and average valuations coincide: the value of an additional unit of installed capital equals the firm's market value per unit of next-period capital.
- ▷ This equality depends also on the model's assumptions and timing.

When Do Investment and Stock Returns Coincide?

- Marginal q vs. average Q : firm's optimal investment vs. valuation of the firm as a whole.
 - Marginal q_t : The value of an additional unit of capital installed at t , available for production at $t + 1$.
 - Average $Q_t = P_t/K_{t+1}$: The firm's post-dividend market value per unit of next-period capital.
- **When do they coincide?** Under constant returns to scale, competitive pricing, and a single scalable capital stock, we established that

$$\underbrace{q_t = P_t/K_{t+1}}_{\text{marginal } q_t = \text{average } Q_t} \quad \underbrace{V_{K,t+1} = V_{t+1}/K_{t+1}}_{\text{marginal value} = \text{average value}}$$

Thus, the marginal value of capital equals its average contribution to firm value.

- The firm's stock return: consider the same homogeneous, all-equity firm. Its gross stock return is

$$R_{t+1}^{S,g} = \frac{P_{t+1} + D_{t+1}}{P_t} = \frac{V_{t+1}/K_{t+1}}{P_t/K_{t+1}} \stackrel{\text{ConRetS}}{=} \frac{V_{K,t+1}}{q_t} = \frac{F_{K,t+1} - G_{K,t+1} + (1 - \delta)q_{t+1}}{q_t} = R_{t+1}^{I,g}.$$

- ↪ Under homogeneity, the average valuations entering the stock return equal the marginal valuations determining the physical investment return. Hence, the two returns coincide **state by state**, not merely in expectation.
- ▷ **When does the distinction matter?** W/out the assumptions, marginal q and average Q need not coincide, neither do their associated returns (e.g., fixed costs, non-scalable rents, multiple stocks, financial leverage,...).

Production-Based Risk Premia

- Recall the investment Euler equation, using the definition of conditional covariance:

$$1 = \mathbb{E}_t[M_{t+1}]\mathbb{E}_t[R_{t+1}^{I,g}] + \text{Cov}_t(M_{t+1}, R_{t+1}^{I,g}).$$

- The investment risk premium:** Since the gross risk-free return satisfies $R_{f,t}^g = 1/\mathbb{E}_t[M_{t+1}]$, rearranging gives:

$$\mathbb{E}_t[R_{t+1}^{I,g}] - R_{f,t}^g = -\frac{\text{Cov}_t(M_{t+1}, R_{t+1}^{I,g})}{\mathbb{E}_t[M_{t+1}]}.$$

- What does production add?* The firm provides an economic characterization of the return being priced:

$$R_{t+1}^{I,g} = \frac{V_{K,t+1}}{q_t}, \quad V_{K,t+1} = F_{K,t+1} - G_{K,t+1} + (1 - \delta)q_{t+1}.$$

Since q_t is known at date t , the risk premium can be expressed directly in terms of the marginal payoff of productive capital:

$$\mathbb{E}_t[R_{t+1}^{I,g}] - R_{f,t}^g = -\frac{\text{Cov}_t(M_{t+1}, V_{K,t+1})}{q_t \mathbb{E}_t[M_{t+1}]}.$$

- **Economic interpretation:** Physical investment commands a positive risk premium when its marginal payoff tends to be low in states where the SDF is high, i.e., when households value additional consumption the most.
- ▷ Production determines the marginal payoff through output, adjustment costs, and the continuation value of capital. The SDF determines how the risk associated with that payoff is priced.

From Technology to Firm Characteristics

- Can we characterize physical investment returns using observable firm characteristics, such as profitability and investment rates?

- Consider a production function with constant returns to scale: $Y_{i,t} = Z_{i,t} K_{i,t}^\alpha L_{i,t}^{1-\alpha}$, $0 < \alpha < 1$.
Differentiating wrt capital:

$$F_{K,i,t} = \alpha Z_{i,t} K_{i,t}^{\alpha-1} L_{i,t}^{1-\alpha} = \alpha Y_{i,t} / K_{i,t}.$$

- Investment and adjustment costs. Under our quadratic adjustment-cost specification: ($x_{i,t} = I_{i,t} / K_{i,t}$)

$$q_{i,t} = 1 + \kappa(x_{i,t} - \delta), \quad -G_{K,i,t} = \frac{\kappa}{2}(x_{i,t}^2 - \delta^2).$$

- Substituting these expressions into the return on marginal capital:

$$R_{i,t+1}^{I,g} = \frac{F_{K,i,t+1} - G_{K,i,t+1} + (1 - \delta)q_{i,t+1}}{q_{i,t}},$$

$$R_{i,t+1}^{I,g} = \frac{\alpha Y_{i,t+1} / K_{i,t+1} + \frac{\kappa}{2}(x_{i,t+1}^2 - \delta^2) + (1 - \delta)[1 + \kappa(x_{i,t+1} - \delta)]}{1 + \kappa(x_{i,t} - \delta)}.$$

- How to read: i) Current investment $x_{i,t}$ determines the marginal cost of installed capital through $q_{i,t}$.
ii) Future profitability/investment determine the marginal payoff through output, adjustment costs, and continuation value.
- ⊲ Under the homogeneity assumptions, the investment return also equals the firm's stock return. However, these firm characteristics do not determine the risk premium on their own (i.e., the SDF).

Investment, Profitability and Expected Returns

- *How do investment and profitability relate to expected returns?*

Investment return is determined by the marginal payoff of capital relative to its marginal cost:

$$(x_{i,t} = I_{i,t}/K_{i,t})$$

$$\mathbb{E}_t[R_{i,t+1}^{I,g}] = \frac{\mathbb{E}_t[V_{K,i,t+1}]}{q_{i,t}} = \frac{\mathbb{E}_t[V_{K,i,t+1}]}{1 + \kappa(x_{i,t} - \delta)}.$$

- Holding the expected marginal payoff $\mathbb{E}_t[V_{K,i,t+1}] > 0$ fixed, differentiating wrt the current investment rate:

$$\frac{\partial \mathbb{E}_t[R_{i,t+1}^{I,g}]}{\partial x_{i,t}} = -\frac{\kappa \mathbb{E}_t[V_{K,i,t+1}]}{[1 + \kappa(x_{i,t} - \delta)]^2} < 0.$$

Higher investment implies a higher marginal cost $q_{i,t}$ and, for a given expected payoff, a lower expected return.

- Holding current investment and the other components of the marginal payoff fixed:

$$\frac{\partial \mathbb{E}_t[R_{i,t+1}^{I,g}]}{\partial \mathbb{E}_t[V_{K,i,t+1}]} = \frac{1}{q_{i,t}} > 0.$$

Higher expected marginal profitability increases the expected return for a given marginal cost of investment.

- For a given expected marginal payoff, a lower ($\downarrow$) required return raises ($\uparrow$) the valuation of installed capital and encourages investment. Conversely, high ($\uparrow$) expected profitability combined with low investment implies a high ($\uparrow$) expected return.

Asset Pricing: Household Demand and Firm Supply

- Two perspectives, one expected return:

Households: Valuation of risk

$$\mathbb{E}_t[R_{t+1}^g] - R_{f,t}^g = -\text{Cov}_t(M_{t+1}, R_{t+1}^g)/\mathbb{E}_t[M_{t+1}].$$

- The SDF reflects households' marginal valuation of consumption across dates and states.
- Expected excess returns compensate for the covariance of payoffs with marginal utility.
- Given the SDF and a return, its risk premium follows from the pricing equation.

- Both conditions must hold for the same investment return:

$$\frac{\mathbb{E}_t[V_{K,t+1}]}{q_t} - R_{f,t}^g = -\frac{\text{Cov}_t(M_{t+1}, V_{K,t+1})}{q_t \mathbb{E}_t[M_{t+1}]}.$$

↪ exp. profitability and investment are linked to the return, reflecting the risk of the marginal payoff.

↪ Production-based asset pricing connects firms' investment and profitability to expected returns.

⊇ The **investment CAPM** refers to the investment-based restrictions on expected returns.

Firms: Production and investment

$$\mathbb{E}_t[R_{t+1}^{I,g}] = \mathbb{E}_t[V_{K,t+1}]/q_t.$$

- Production technology determines the marginal payoff of productive capital.
- Firm optimality links the marginal cost q_t to the investment rate.
- Given expected marginal payoffs, investment decisions restrict expected returns.

From Production-Based Pricing to Factor Models

- We have shown how production and investment decisions determine physical investment returns.
 $\hookrightarrow$ *Can these returns help explain risk premia across financial assets?*
- **A linear projection of the SDF:** consider a vector of investment-related factors f_{t+1} , such as physical investment returns or innovations in investment opportunities. We can write:

$$M_{t+1} = a_t + b'_t f_{t+1} + \varepsilon_{t+1}, \quad \mathbb{E}_t[\varepsilon_{t+1}] = 0, \quad \text{Cov}_t(\varepsilon_{t+1}, f_{t+1}) = 0.$$

The projection separates the variation in the SDF explained by the selected factors from the ε_{t+1} .

$\hookrightarrow$ Substituting the projection into the SDF:

$$\mathbb{E}_t[R_{i,t+1}^g] - R_{f,t}^g = - \frac{b'_t \text{Cov}_t(f_{t+1}, R_{i,t+1}^g) + \text{Cov}_t(\varepsilon_{t+1}, R_{i,t+1}^g)}{\mathbb{E}_t[M_{t+1}]}.$$

- When does this become a factor-pricing model?
 - If $\text{Cov}_t(\varepsilon_{t+1}, R_{i,t+1}^g) = 0$ for all assets, risk premia depend only on their covariances with the factors.
 - Orthogonality between ε_{t+1} and f_{t+1} alone does not guarantee this condition.
 - **An investment-return factor:** For example, if $M_{t+1} = a_t - b_t R_{t+1}^{I,g} + \varepsilon_{t+1}$, the factor covariance coefficient in the expected excess return is $b_t / \mathbb{E}_t[M_{t+1}]$, provided the residual is unpriced. Neither its sign nor its magnitude follows from firm optimality alone.
- $\hookrightarrow$ **An important distinction:** Current q_t and I_t/K_t are known at date t and cannot themselves explain conditional return covariances. Future investment returns and innovations in investment opportunities can serve as risk factors; current firm characteristics can instead help predict expected returns.

TBD

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