Asset Pricing Theory: S02 Consumption Risk, Asset Prices, and the Benchmark Puzzles

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Structure of the course

- Email: amedeo.andriollo@dauphine.psl.eu
- Grading follows a 30-70 rule: 70% final exam, 30% project/homework (4th, 7th sessions).
- Office hours: feel “free” to (DO) come: B612.
- Slides/materials are updated regularly. Report a typo/mistake: corrections help the class.
- This session:
  - Binary Economy, CARA and CRRA with consumption, Puzzles, Rare disasters model, Campbell-Cochrane ’99.
  - Main reference: Campbell (2018, Chs. 4 and 6); Cochrane (2005, Chs. 1, 2, and 21).

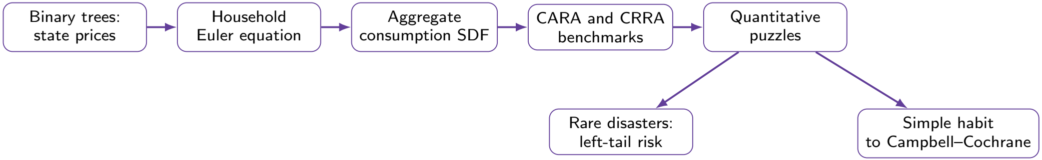
▷ Next session: Dynamic Asset Pricing.

Today's Route: From a Pricing Kernel to Economic Mechanisms

- Session 1 moved from No Arbitratge to a positive pricing kernel:

$$P_{i,t} = \mathbb{E}_t[M_{t+1}X_{i,t+1}], \qquad 1 = \mathbb{E}_t[M_{t+1}R^g_{i,t+1}].$$

- This session asks what moves M from the lens of aggregate consumption. What we see? The consumption model struggles quantitatively.



- ▷ Each extension still passes through $1 = \mathbb{E}_t[M_{t+1}R^g_{i,t+1}]$.
- ▷ We return to intuition after each derivation and confront the benchmark with data.

Binary Economy: Setup and No-Arbitrage Region

- Two dates and two states: today and tomorrow; good G with prob. π, bad B with prob. $1-\pi$.
- The bond costs 1 today and pays the gross safe return $R_f^g \equiv 1 + r_f$ tomorrow (no matter the state).
- The stock costs S and pays uS (dS) in G (B), with $u > d > 0$.
- No arbitrage requires $d < R_f^g < u$: bond and stock should not dominate one another, state by state.

↪ Why? If, for example, $R_f^g \leq d$, buy one share of stock and short S bonds. The portfolio costs zero today and pays in good state $S(u - R_f^g) > 0$ and in bad state $S(d - R_f^g) \geq 0$. If instead $R_f^g \geq u$, reverse the positions: buy S bonds and short one share of stock.

- Consider a claim X, paying X_u in G and X_d in B. The replicating portfolio with b bonds and θ stock shares would be:

$$bR_f^g + \theta uS = X_u, \qquad bR_f^g + \theta dS = X_d.$$

↪ Session 1 guarantees a price; Session 2 asks why bad-state payoffs receive special weight.

Binary Economy: Replicate Any Claim

- Subtracting the two payoff equations gives

$$\theta = \frac{X_u - X_d}{(u-d)S}.$$

- Substitution gives

$$b = \frac{uX_d - dX_u}{(u-d)R_f^g}.$$

- By the Law of One Price, $p = b + \theta S$, hence

$$\begin{aligned} p &= \frac{uX_d - dX_u}{(u-d)R_f^g} + \frac{X_u - X_d}{u-d} = \frac{1}{R_f^g}\left[\frac{R_f^g - d}{u-d}X_u + \frac{u - R_f^g}{u-d}X_d\right] \\ &= \frac{1}{R_f^g}\left[\widetilde{q}X_u + (1-\widetilde{q})X_d\right] = \frac{1}{R_f^g}\mathbb{E}^{\mathbb{Q}}[X], \end{aligned}$$

where $\widetilde{q} = (R_f^g - d)/(u-d) \in (0,1)$.

↪ The no-arbitrage restriction $d < R_f^g < u$ makes both state weights strictly positive: $R_f^g > d \Rightarrow \widetilde{q} > 0, R_f^g < u \Rightarrow 1 - \widetilde{q} > 0$. Hence $\widetilde{q} \in (0,1)$: the no-arbitrage condition is exactly what makes these weights valid risk-neutral probabilities.

Binary Economy: State Prices Become an SDF

- Arrow-Debreu prices are

$$q_G = \frac{\widetilde{q}}{R_f^g}, \qquad q_B = \frac{1-\widetilde{q}}{R_f^g}, \qquad (q_G + q_B)R_f^g = 1.$$

$\hookrightarrow$ where the last one is due to the definition of Arrow-Debreu state prices: q_B (q_G) is today's price of a claim paying 1 in bad (good) state.

- Divide state prices by physical probabilities:

$$M_G = \frac{q_G}{\pi} = \frac{\widetilde{q}}{\pi R_f^g}, \qquad M_B = \frac{q_B}{1-\pi} = \frac{1-\widetilde{q}}{(1-\pi)R_f^g}.$$

- Therefore

$$\boxed{p = \pi M_G X_u + (1-\pi) M_B X_d = \mathbb{E}[MX].}$$

- No arbitrage determines a positive M, but not what makes M_B larger or smaller than M_G.

Pricing Rule vs. Economic Driver

Relative pricing

$$P_i = \mathbb{E}[MX_i]$$

- No arbitrage restricts prices.
- Covariance with M restricts expected returns.
- Open question: What moves M?

Equilibrium candidate

$$M^j = \beta \frac{u_j'(c_{j,1})}{u_j'(c_{j,0})}$$

- From investor j's consumption.
- Aggregate consumption needs extra assumptions.
- Open question: How stringent are the assumptions?

What turns an abstract pricing kernel into a consumption model?

Consumption-Based Pricing: Preferences and Choices

- An investor j with wealth W_0 buys i) θ units of risky payoff X at price p, and ii) b bonds at price 1.
- Preferences are over consumption at dates 0 (today) and 1 (tomorrow).
- Preferences over lotteries that are i) complete, ii) transitive, iii) continuous, and iv) independent admit a (expected) utility representation.
  $\hookrightarrow$ the utility function is unique up to positive affine transformations.
- With $0 < \beta < 1$ (subjective discount), the investor evaluates

$$u(C_0) + \beta \mathbb{E}[u(C_1)].$$

$\hookrightarrow$ Saving and portfolio choice transfer consumption across dates and states.

The Investor's Problem Produces the Consumption SDF

- Using the constraints, consumption follows

$$C_0 = W_0 - p\theta - b, \qquad C_1 = R_f^g b + \theta X.$$

- The investor solves the unconstrained:

$$\max_{\theta,b} \; u(W_0 - p\theta - b) + \beta\mathbb{E}[u(R_f^g b + \theta X)].$$

- By FOCs, an interior solution (w.r.t. b and θ):

$$u'(C_0) = \beta R_f^g \mathbb{E}[u'(C_1)], \qquad p\,u'(C_0) = \beta\mathbb{E}[X u'(C_1)].$$

$\hookrightarrow$ Euler equation compares the utility cost of giving up a little consumption today with the utility benefit of the payoff the (risky) asset delivers tomorrow (state by state).

- Thus, solving for p:

$$\boxed{p = \mathbb{E}[MX]}, \qquad \boxed{M = \beta\frac{u'(C_1)}{u'(C_0)}}.$$

From Individual to Aggregate

- From the individual investor's problem, investor j has the SDF: $M^j_{t+1} = \beta \frac{u'_j(c_{j,t+1})}{u'_j(c_{j,t})}$.
  $\hookrightarrow$ individual consumption.
- Moving from $c_{j,t}$ to aggregate consumption C_t requires equilibrium structure: risk sharing must align marginal utilities across investors, and market clearing must aggregate individual choices.
- Under a representative-agent formulation, the common pricing kernel can then be written directly as

$$M_{t+1} = \beta \frac{u'(C_{t+1})}{u'(C_t)}.$$

- In a pure endowment economy, market clearing further implies total consumption equals total production: $C_t = Y_t$, so preferences and the aggregate endowment process jointly determine consumption and asset prices.

$\hookrightarrow$ Replacing individual consumption c_j with aggregate consumption C is an **equilibrium restriction**.

$\trianglerighteq$ We can now ask which payoffs are especially valuable when aggregate consumption is low.

Bad-State Payoffs Are Consumption Insurance

Bad consumption state

C_{t+1} low

$\Downarrow$

$u'(C_{t+1})$ high

$\Downarrow$

M_{t+1} high

A claim pays in that state

$\mathrm{Cov}_t(M, X) > 0$

$\Downarrow$

$P = \mathbb{E}_t[MX]$ high

$\Downarrow$

expected return low

- Volatility ("spread" in payoff) alone is not priced risk: what matters is when the payoff arrives.
- Investors are willing to pay more for assets that pay off in bad times when consumption is low (and less for assets that pay off in good times when consumption is high).

$\hookrightarrow$ Assets that fail when marginal utility is high provide poor insurance and must offer higher exp. returns.

CRRA Maps Consumption Growth into the SDF

- Constant Relative Risk Aversion, including the limiting log-utility case:

$$u(C) = \begin{cases} \dfrac{C^{1-\gamma} - 1}{1-\gamma}, & \gamma \neq 1, \\ \log C, & \gamma = 1, \end{cases} \qquad u'(C) = C^{-\gamma} = 1/C^{\gamma}.$$

- Define $G_{t+1} = C_{t+1}/C_t$ and $\Delta c_{t+1} = \log G_{t+1}$.
- Then: $M_{t+1} = \beta G_{t+1}^{-\gamma}$, $m_{t+1} = \log \beta - \gamma \Delta c_{t+1}$.

$\hookrightarrow$ (Aggregate) consumption growth is the primitive driving variation in the benchmark SDF:

- higher (lower) consumption growth means a lower (higher) SDF;
- an extra dollar delivered in a good consumption state is not especially valuable.

In-Class Exercise: Price the Timing of Cash Flows

- Two equally likely states and CRRA preferences with $\beta = 0.96, \gamma = 2$:

	Good	Bad
$\pi(s)$	0.5	0.5
G_{t+1}	1.04	0.96

- Two claims have the same expected payoff and dispersion:

	Good	Bad	$\mathbb{E}[X]$
X^C (cyclical)	1.5	0.5	1
X^I (insurance)	0.5	1.5	1

- Now, take 15' to:
  1. Compute M_g and M_b from $M_s = \beta G_s^{-\gamma}$.
  2. Compute $R_f^g = 1/\mathbb{E}[M]$.
  3. Price each claim:
  $$P^k = \tfrac{1}{2} M_g X_g^k + \tfrac{1}{2} M_b X_b^k, \qquad k \in \{C, I\}.$$
  4. Since $\mathbb{E}[X^k] = 1$, compute $\mathbb{E}[R_k^g] = 1/P^k$.
  5. Provide the economic explanation using $\mathrm{Cov}(M, X^k)$.

Solution: The SDF Is High in the Bad State

- State-contingent SDF values are

$$M_g = 0.96(1.04)^{-2} = 0.8876, \qquad M_b = 0.96(0.96)^{-2} = 1.0417.$$

- Therefore: $\mathbb{E}[M] = \frac{1}{2}(0.8876 + 1.0417) = 0.9646 \implies R_f^g = 1/0.9646 = 1.0367.$
  $\hookrightarrow$ The safe net return is something around 3.7%.
- One unit of payoff is more valuable in the bad state because $M_b > M_g$.

	Cyclical X^C	Insurance X^I
Expected payoff	1.0000	1.0000
Price $P = \mathbb{E}[MX]$	0.9261	1.0031
Expected gross return $\mathbb{E}[R^g] = \frac{\mathbb{E}X}{P}$	1.0798	0.9969
$\mathrm{Cov}(M, X) = \mathbb{E}[MX] - \mathbb{E}[M]\mathbb{E}[X]$	-ive (-3%)	+ive ($+3\%$)

- Mean cash flow and payoff volatility cannot explain the price difference, while state/risk alignment (i.e., Arrow-Debreu world) does: the insurance claim pays when marginal utility is high.

$\hookrightarrow$ Consumption-timing risk, not volatility by itself, determines the premium.

CARA-Normal utility

- CARA (exponential) utility is $u(C) = -e^{-\gamma C}$, with Constant Absolute Risk Aversion $\gamma > 0$.
- If $X \sim \mathcal{N}(\mu, \sigma^2)$,
$$\mathbb{E}[u(X)] = \mathbb{E}[-e^{-\gamma X}] = -\exp\left(-\gamma\mu + \tfrac{1}{2}\gamma^2\sigma^2\right).$$

$\hookrightarrow$ CARA plus normality turns portfolio choice into an **exact** mean–variance problem.

- Consider the previous investor problem: $C_0 = W_0 - p\theta - b$, $C_1 = R_f^g b + \theta X$, with now $X \sim \mathcal{N}(\mu, \sigma^2)$.
- The utility max objective becomes:

$$\max_{b,\theta} -e^{-\gamma(W_0 - p\theta - b)} + \beta\mathbb{E}[-e^{-\gamma(R_f^g b + \theta X)}].$$

- CARA+Normality gives:

$$\mathbb{E}[-e^{-\gamma(R_f^g b + \theta X)}] = -\exp\left[-\gamma(R_f^g b + \theta\mu) + \tfrac{1}{2}\gamma^2\theta^2\sigma^2\right].$$

- It isolates expected payoff, variance, risk aversion (notice that risk enters through $\theta^2\sigma^2$).
- Drawbacks? 1) normal payoffs are unbounded, and risky dollar demand is wealth-invariant, see the next slide.

From Optimal Demand to Equilibrium Price

- Let $A = -\gamma(R_f^g b + \theta\mu) + \frac{1}{2}\gamma^2\theta^2\sigma^2$. The bond and risky-asset FOCs are (w.r.t. b and θ)

$$e^{-\gamma C_0} = \beta R_f^g e^A, \qquad -\gamma p e^{-\gamma C_0} + \beta e^A(\gamma\mu - \gamma^2\theta\sigma^2) = 0.$$

- Substituting the former into the latter:

$$\mu - R_f^g p - \gamma\theta\sigma^2 = 0,$$

  and therefore optimal risky demand is

$$\theta^*(p) = \frac{\mu - R_f^g p}{\gamma\sigma^2}.$$

- With outside supply $\bar{\theta}$, market clearing requires $\theta^*(p) = \bar{\theta}$, so the equilibrium price is $p = \frac{\mu - \gamma\bar{\theta}\sigma^2}{R_f^g}$.
- Price is the discounted expected payoff minus a risk discount: greater risk aversion, payoff variance, or risky-asset supply lowers the equilibrium price.

↪ CARA–normality delivers this relation in closed form.

From the Euler Equation to Lognormal Restrictions

- For any gross return $R^g_{i,t+1}$, the fundamental pricing restriction is $1 = \mathbb{E}_t[M_{t+1} R^g_{i,t+1}]$. For the risk-free asset, $R^g_{f,t} = 1/\mathbb{E}_t[M_{t+1}]$.
- Define log SDF and log return:

$$m_{t+1} \equiv \ln M_{t+1}, \qquad r_{i,t+1} \equiv \ln R^g_{i,t+1}.$$

so that the Euler rewrites as: $0 = \ln \mathbb{E}_t[e^{m_{t+1}+r_{i,t+1}}]$

! **Assumption:** suppose that, conditionally on date-t information, $(m_{t+1}, r_{i,t+1})$ is jointly normal.

- Since, for conditionally normal z, we have:

$$\ln \mathbb{E}_t[e^z] = \mathbb{E}_t[z] + \tfrac{1}{2}\operatorname{Var}_t(z),$$

then the Euler equation (once subtituting $M_t = \beta$) becomes

$$\boxed{0 = \mathbb{E}_t[m_{t+1} + r_{i,t+1}] + \tfrac{1}{2}\operatorname{Var}_t(m_{t+1} + r_{i,t+1}).}$$

- **Exact under joint normality**; otherwise it is a second-order cumulant approximation.
- The lognormal Euler equation expands to: $0 = \mathbb{E}_t[m] + \mathbb{E}_t[r_i] + \frac{1}{2}\operatorname{Var}_t(m) + \frac{1}{2}\operatorname{Var}_t(r_i) + \operatorname{Cov}_t(m, r_i)$.

▷ We can now combine this restriction with the CRRA SDF to study the safe rate and risky premia.

Lognormal CRRA: Three Forces in the Safe Rate

- For the $r_{f,t}$, since it is known at time t: $0 = \mathbb{E}_t[m] + r_{f,t} + \frac{1}{2}\operatorname{Var}_t(m)$.
  $\hookrightarrow$ Under CRRA, $m = \log\beta - \gamma\Delta c$. If consumption growth and returns are conditionally normal,

$$r_{f,t} = -\log\beta + \gamma\mathbb{E}_t[\Delta c_{t+1}] - \tfrac{1}{2}\gamma^2 \operatorname{Var}_t(\Delta c_{t+1}).$$

  3 forces: $-\log\beta$: impatience; $\gamma\mathbb{E}_t[\Delta c]$: intertemporal smoothing; $-\gamma^2\sigma_c^2/2$: precautionary saving. (The effect of γ is not globally monotone).
- Subtracting the safe-return equation from the risky one,

$$\mathbb{E}_t[r_i] - r_{f,t} + \tfrac{1}{2}\operatorname{Var}_t(r_i) = -\operatorname{Cov}_t(m, r_i).$$

- Under CRRA,

$$\boxed{\mathbb{E}_t[r_i] - r_{f,t} + \tfrac{1}{2}\operatorname{Var}_t(r_i) = \gamma\operatorname{Cov}_t(\Delta c, r_i).}$$

- Equivalently under lognormality (as implies $\log\mathbb{E}_t[R_i^g] = \mathbb{E}_t[r_i] + \frac{1}{2}\operatorname{Var}_t(r_i)$),

$$\log\mathbb{E}_t[R_i^g] - r_{f,t} = \gamma\operatorname{Cov}_t(\Delta c, r_i).$$

- The half-variance term is the Jensen adjustment: $R_i^g = e^{r_i}$, not $1 + r_i$.

$\hookrightarrow$ Log methods convert a nonlinear Euler equation into restrictions on means, variances, and covariances.

One CRRA SDF Must Price Safe and Risky Assets Jointly

- Under CRRA and conditional lognormality, the same SDF, $m_{t+1} = \log\beta - \gamma\Delta c_{t+1}$, must satisfy two distinct pricing restrictions.
- For the risk-free asset,

$$r_{f,t} = -\ln\beta + \gamma\mathbb{E}_t[\Delta c_{t+1}] - \tfrac{1}{2}\gamma^2 \operatorname{Var}_t(\Delta c_{t+1}).$$

  $\hookrightarrow$ the safe rate is disciplined by: i) time preference, ii) expected consumption growth (mean of m), and iii) consumption-growth uncertainty (variance of m).
- For a risky asset,

$$\ln\mathbb{E}_t[R_i^g] - r_{f,t} = \gamma \operatorname{Cov}_t(\Delta c_{t+1}, r_{i,t+1}).$$

  Risk compensation is disciplined by the asset's covariance with consumption growth and by the curvature γ.

! γ needs to match risky premia and the model-implied safe rate.

▷ We now confront these joint restrictions with historical U.S. consumption and return data.

Equity-Premium Puzzle: Too little Variation

Annualized percent	Equity log return	Real short rate	Consumption growth
Mean	6.85	0.72	1.74
Standard deviation	15.98	1.78	1.64

- U.S. quarterly sample, 1947Q2–2011Q2; real annualized log variables; per-capita consumption growth.
- Contemporaneous equity-return/consumption-growth correlation: 0.18. Annualized arithmetic premium: (volatility in pp) $\ln \mathbb{E}_t[R_i^g] - r_{f,t} = 6.85 - 0.72 + \frac{1}{2}(15.86^2)/100 = 7.39\%$.
- Consumption growth is smooth (little variation) and weakly correlated with equity returns.
- The lognormal CRRA restriction is: $\log\left(\frac{\mathbb{E}[R_e^g]}{R_f^g}\right) = \gamma\,\mathrm{Cov}(\Delta c, r_e)$.

  $\hookrightarrow$ Using the sample covariance requires approximately $\gamma \approx 155$.

  $\hookrightarrow$ Even imposing the max correlation gives a lower bound near $\gamma \approx 28$: extreme aversion.
- Consumption is too smooth and too weakly correlated with returns to price the premium.

Three Quantitative Failures of the CRRA Benchmark

- The equity-premium reveals one problem, but the same SDF must account for i) the safe rate and ii) the behavior of asset prices over time.

Puzzle	What the benchmark implies	What appears to be missing
Equity premium	$\log \mathbb{E}[R_e^g] - r_f = \gamma \operatorname{Cov}(\Delta c, r_e) \implies$ Smooth consumption and weak covariance require very large γ to match the historical premium.	More priced bad-state exposure, or a stronger state-contingent price of risk.
Risk-free rate	$r_f = -\log \beta + \gamma \mu_c - \frac{1}{2}\gamma^2\sigma_c^2 \implies$ The same large γ used to raise the equity premium also changes the model-implied safe rate.	A mechanism that separates risk compensation from intertemporal smoothing and precautionary saving.
Equity volatility	$m_{t+1} = \log \beta - \gamma \Delta c_{t+1} \implies$ Smooth consumption generates a relatively smooth SDF, while returns, valuation ratios, and expected returns vary substantially.	State-dependent SDF volatility and a time-varying price of risk.

▷ We therefore consider richer economic mechanisms: habit formation and rare-disaster risk.

Simple Growth-Based Habit:

- Consider the tractable external-habit specification

$$u(C_t, H_t) = \frac{1}{1-\gamma} C_t^{1-\gamma} \left(\frac{H_t}{C_t} \right)^{\kappa} = \frac{C_t^{1-\gamma-\kappa} H_t^{\kappa}}{1-\gamma},$$

  with $H_t = C_{t-1}$, $\gamma > 1$, $\kappa \geq 0$. The household takes the lagged aggregate habit H_t as given.
- Habit makes marginal utility more sensitive to changes in consumption:
  - Marginal utility is $u_C(C_t, H_t) = \frac{1-\gamma-\kappa}{1-\gamma} C_t^{-\gamma-\kappa} H_t^{\kappa}$.
  - The constant cancels from the intertemporal ratio: $M_{t+1} = \beta \left(\frac{C_{t+1}}{C_t} \right)^{-(\gamma+\kappa)} \left(\frac{C_t}{C_{t-1}} \right)^{\kappa}$.

  $\hookrightarrow$ We have:

$$m_{t+1} = \log \beta - (\gamma + \kappa) \Delta c_{t+1} + \kappa \Delta c_t.$$

  The log SDF will load on next-period consumption growth $-(\gamma + \kappa)\Delta c_{t+1}$ rather than $-\gamma \Delta c_{t+1}$.
- For $\kappa > 0$, the same negative consumption-growth shock produces a larger increase in marginal utility and SDF: $\Delta c_{t+1} \downarrow \implies M_{t+1} \uparrow$ more strongly than under no-habit CRRA.

Simple Habit: What It Achieves—and What It Misses

- The unexpected component is $\widetilde{m}_{t+1} = -(\gamma + \kappa)\widetilde{\Delta c}_{t+1}$.
- From $m_{t+1} = \log\beta - (\gamma+\kappa)\Delta c_{t+1} + \kappa\Delta c_t$, and conditional normality of Δc_{t+1},

$$\boxed{r_{f,t} = -\log\beta - \kappa\Delta c_t + (\gamma+\kappa)\mu_{c,t} - \tfrac{1}{2}(\gamma+\kappa)^2\sigma^2_{c,t}.}$$

- This amplification can raise risk premia for a given γ:
  bad consumption-growth realizations receive greater marginal-utility weight.
  However:
  - The loading $\gamma + \kappa$ is constant across states: every consumption shock is amplified by the same amount.
  - the term $+\kappa\Delta c_t$ does not contribute to the unexpected variation: it moves the conditional safe rate.
  - $\hookrightarrow$ stronger shock sensitivity alone does not jointly deliver a low, smooth safe rate and persistent, countercyclical risk premia.

$\trianglerighteq$ Simple habit is therefore a stepping stone: richer models must make the price of risk itself state dependent.

Rare Disasters: Setup and Safe Rate

- Low-probability, high-severity consumption drops follow Rietz (1988) and Barro (2006).
- In a one-period representative-agent endowment economy,

$$\Delta c = \mu_c + \sigma_c \varepsilon - J \cdot D, \quad \varepsilon \sim N(0,1), \quad D \sim \text{Bernoulli}(p), \quad J > 0, p \ll 1.$$

  $\hookrightarrow$ With probability p, consumption falls by fraction $1 - e^{-J}$
  (e.g., wars, famines, pandemics, financial crises, and natural disasters).
- Portfolios chosen today; shocks and payoffs realize tomorrow; aggregate consumption equals endowment.
- Under CRRA, we have: $M = \beta(C_1/C_0)^{-\gamma} = \beta e^{-\gamma \Delta c}$
  $\hookrightarrow$ if we substitute, a disaster multiplies marginal utility by $e^{\gamma J}$.
- Normal volatility and tail risk are distinct: a rare state matters greatly because marginal utility is nonlinear.
- Independence lets the SDF expectation factor:

$$\mathbb{E}[M] = \beta e^{-\gamma \mu_c} \mathbb{E}[e^{-\gamma \sigma_c \varepsilon}] \mathbb{E}[e^{\gamma J D}] = \beta e^{-\gamma \mu_c + \frac{1}{2}\gamma^2 \sigma_c^2} \left[(1-p)e^{\gamma J \cdot 0} + p e^{\gamma J}\right].$$

- Since $R_f^g = 1/\mathbb{E}[M]$,

$$\boxed{r_f = -\log \beta + \gamma \mu_c - \frac{1}{2}\gamma^2 \sigma_c^2 - \log\left[(1-p) + p e^{\gamma J}\right].}$$

- Disaster states have high marginal utility: higher price of safe consumption, lower safe rate.

Rare Disasters: Asset Exposure and Pricing

- Define a generic asset's log return: $r \equiv \log R^g = a + k\Delta c + \eta$, with $\eta \perp (\varepsilon, D)$ and $\eta \sim N(0, s^2)$. The coefficient k measures the asset's exposure to both consumption shocks and disasters.
-
- Since the Euler equation $\mathbb{E}[MR^g] = \mathbb{E}[e^{m+r}] = 1$ and $m = \log\beta - \gamma\Delta c$, we have

$$\begin{aligned} m + r &= \log\beta + a + (k-\gamma)\Delta c + \eta \\ &= \log\beta + a + (k-\gamma)\mu_c + (k-\gamma)\sigma_c\varepsilon - (k-\gamma)JD + \eta. \end{aligned}$$

- therefore factors into

$$\begin{aligned} 1 &= \beta e^{\,a+(k-\gamma)\mu_c}\,\mathbb{E}\Big[e^{(k-\gamma)\sigma_c\varepsilon+\eta}\Big]\,\mathbb{E}\Big[e^{-(k-\gamma)JD}\Big] \\ &= \beta e^{\,a+(k-\gamma)\mu_c+\frac{1}{2}[(k-\gamma)^2\sigma_c^2+s^2]}\Big[(1-p)+pe^{-(k-\gamma)J}\Big]. \end{aligned}$$

- Taking logs and solving for the return intercept gives

$$\boxed{\begin{aligned} a = &-\log\beta - (k-\gamma)\mu_c - \tfrac{1}{2}[(k-\gamma)^2\sigma_c^2 + s^2] \\ &-\log\Big[(1-p)+pe^{-(k-\gamma)J}\Big]. \end{aligned}}$$

$\hookrightarrow$ Pricing therefore adjusts the asset's expected return for both its ordinary consumption exposure and its disaster exposure.

Rare Disasters: Exact Bernoulli Risk Premium

- Define $A(zJ) \equiv (1-p) + pe^{zJ}$.

$\hookrightarrow$ $a = -\log\beta - (k-\gamma)\mu_c - \frac{1}{2}[(k-\gamma)^2\sigma_c^2 + s^2] - \log A((\gamma-k)J)$

- Since $r = a + k\Delta c + \eta$,

$$\log \mathbb{E}[R^g] = a + k\mu_c + \frac{1}{2}(k^2\sigma_c^2 + s^2) + \log A(-kJ).$$

$\hookrightarrow$ $r_f = -\log\beta + \gamma\mu_c - \frac{1}{2}\gamma^2\sigma_c^2 - \log A(\gamma J)$.

- Combining this expression with the pricing intercept and the safe rate,

$$\boxed{\log \mathbb{E}[R^g] - r_f = k\gamma\sigma_c^2 + \log\frac{A(\gamma J)A(-kJ)}{A((\gamma-k)J)}.}$$

- For any α,

$$\log[A(\alpha)] = \log[(1-p) + pe^{\alpha}] = \log[1 + p(e^{\alpha}-1)] = p(e^{\alpha}-1) + \mathcal{O}(p^2(e^{\alpha}-1)^2)$$

- Applying the expansion to all three Bernoulli cumulants,

$$\log \mathbb{E}[R^g] - r_f \approx k\gamma\sigma_c^2 + p[e^{\gamma J} + e^{-kJ} - e^{(\gamma-k)J} - 1].$$

- Equivalently, the disaster term is $p(e^{\gamma J}-1)(1-e^{-kJ})$.

Rare Disasters: Comparative Statics and the Cross-Section

- **Diffusion risk:** $k\gamma\sigma_c^2$ rises with exposure, curvature, and ordinary consumption variance.
- **Disaster risk:** for $\gamma, k, J > 0$,
$$p(e^{\gamma J} - 1)(1 - e^{-kJ}) > 0.$$
- If severity is also small, $e^{\alpha} - 1 \approx \alpha + \alpha^2/2$; convexity makes larger disasters disproportionately important.
- Small p with sizable J can raise premia without extreme curvature.
- Across assets, higher k means greater disaster/diffusion exposure and a higher premium; in other words, low or negative k identifies hedging assets.

Campbell's Jump Representation Is Complementary

- Campbell's permits a compound-Poisson number of jumps:

$$\Delta c = \mu + \sigma\varepsilon - \sum_{j=1}^{N} x_j, \qquad N \sim \text{Poisson}(\omega).$$

- Its cumulant-generating function is

$$c(\theta) = \mu\theta + \tfrac{1}{2}\sigma^2\theta^2 + \omega\{\mathbb{E}[e^{-\theta x}] - 1\}.$$

- For a consumption claim with retained fraction $B = e^{-x}$,

$$\boxed{\log \mathbb{E}[R_C^g] - \log R_f^g = \gamma\sigma^2 + \omega\mathbb{E}[(1-B)(B^{-\gamma}-1)].}$$

- This corroborates the Bernoulli result.

I.I.D. Disasters Are Not Persistent

- Rare disasters add economically important left-tail consumption risk: even a low-probability disaster receives a large marginal-utility weight, because

$$M = \beta e^{-\gamma \Delta c} \quad \Longrightarrow \quad D = 1 \text{ multiplies the SDF by } e^{\gamma J}.$$

- With constant disaster probability p and severity J, this mechanism can raise the unconditional equity premium and lower the safe rate.
- But if disasters are i.i.d., then the conditional disaster distribution is the same every period:

$$\mathsf{Pr}_t(D_{t+1} = 1) = p \qquad \text{for all } t.$$

  Hence disaster risk does not become systematically larger in bad times or smaller in good times.
- Consequently, the i.i.d. benchmark does not by itself generate persistent movements in expected returns, price–dividend ratios, or SDF volatility.

↪ Rare disasters change the **distribution of consumption risk**.
Habit instead makes the **price of ordinary consumption risk state dependent**.

Campbell–Cochrane '99: Surplus Is the State

- Let X_t be an external habit and define $S_t = \frac{C_t - X_t}{C_t} \in (0,1)$, $s_t = \log S_t$.
- Utility is CRRA over surplus consumption:

$$u(C_t, X_t) = \begin{cases} \dfrac{(C_t - X_t)^{1-\gamma} - 1}{1-\gamma}, & \gamma \neq 1, \\ \log(C_t - X_t), & \gamma = 1. \end{cases}$$

- Marginal utility is $u_C = (C_t - X_t)^{-\gamma} = (S_t C_t)^{-\gamma}$. Relative risk aversion is: $-C_t u''/u' = \gamma / S_t$.
  $\hookrightarrow$ when consumption approaches habit, S_t falls and effective risk aversion rises.
- Therefore

$$\boxed{M_{t+1} = \beta \left(\frac{S_{t+1} C_{t+1}}{S_t C_t} \right)^{-\gamma}.}$$

- In logs, $m_{t+1} = \log \beta - \gamma \Delta c_{t+1} - \gamma \Delta s_{t+1}$.
- The surplus ratio is a **new state variable**: it can move marginal utility even when consumption growth is smooth.

$\hookrightarrow$ countercyclical risk premia with a low and stable safe rate.

Campbell–Cochrane: Canonical State Dynamics

- Consumption growth is i.i.d. normal: $\Delta c_{t+1} = g + \varepsilon_{c,t+1}$, with $\varepsilon_c \sim N(0, \sigma_c^2)$.
- Log surplus consumption follows: $s_{t+1} = (1-\phi)\bar{s} + \phi s_t + \lambda(s_t)\varepsilon_{c,t+1}$.
- The canonical sensitivity function is

$$\lambda(s) = \begin{cases} \frac{1}{\bar{S}}\sqrt{1 - 2(s - \bar{s})} - 1, & s \leq s_{\max}, \\ 0, & s > s_{\max}, \end{cases}$$

with: $\bar{S} = \sigma_c \sqrt{\frac{\gamma}{1-\phi}}$, $s_{\max} = \bar{s} + \frac{1-\bar{S}^2}{2}$.

- Unlike a constant loading, $\lambda(s_t)$ controls the safe rate while preserving bad-times amplification.
- Substituting the state law into the log SDF gives

$$m_{t+1} = \log\beta - \gamma g - \gamma(1-\phi)(\bar{s} - s_t) - \gamma[1 + \lambda(s_t)]\varepsilon_{c,t+1}.$$

- The conditional innovation and volatility are

$$\widetilde{m}_{t+1} = -\gamma[1 + \lambda(s_t)]\varepsilon_{c,t+1}, \qquad \sigma_t(m) = \gamma[1 + \lambda(s_t)]\sigma_c.$$

- Low s_t (bad times) implies high $\lambda(s_t)$, volatile marginal utility, and a high price of consumption risk.
- Despite γ is constant, habit makes effective risk aversion countercyclical.

The Sensitivity Makes the Safe Rate Constant

- Conditional lognormality gives

$$r_{f,t} = -\log\beta + \gamma g - \gamma(1-\phi)(s_t - \bar{s}) - \tfrac{1}{2}\gamma^2[1+\lambda(s_t)]^2\sigma_c^2.$$

- In the nonzero region (not too good times),

$$[1+\lambda(s_t)]^2 = \frac{1-2(s_t-\bar{s})}{\bar{S}^2}, \qquad \bar{S}^2 = \frac{\gamma\sigma_c^2}{1-\phi}.$$

- Substitution makes the state-dependent terms cancel:

$$\boxed{r_f = -\log\beta + \gamma g - \tfrac{1}{2}\gamma(1-\phi).}$$

↪ The mechanism is conditional-mean/conditional-variance offset, not cancellation of the realized shock.

- The canonical calibration delivers a constant safe rate while SDF volatility varies.

Time-Varying Price of Risk

- For every asset, $1 = \mathbb{E}_t[M_{t+1} R^g_{i,t+1}]$. With $m = \log M$ and $r_i = \log R^g_i$, $0 = \log \mathbb{E}_t[e^{m_{t+1}+r_{i,t+1}}]$.
- Under conditional joint normality (or as a second-order approximation)

$$\begin{aligned}\mathbb{E}_t[r_i] &\approx -\mathbb{E}_t[m] - \tfrac{1}{2}\operatorname{Var}_t(m) - \tfrac{1}{2}\operatorname{Var}_t(r_i) - \operatorname{Cov}_t(m, r_i)\\ &= r_{f,t} - \tfrac{1}{2}\operatorname{Var}_t(r_i) - \operatorname{Cov}_t(m, r_i).\end{aligned}$$

- Let asset i have log return

$$r_{i,t+1} = \mu_i + b_i \varepsilon_{c,t+1} + \eta_{i,t+1}, \qquad \eta_i \perp \varepsilon_c, \quad \operatorname{Var}_t(\eta_i) = s_i^2.$$

- The SDF price per unit consumption shock is

$$\Lambda_t \equiv -\frac{\operatorname{Cov}_t(m_{t+1}, \varepsilon_{c,t+1})}{\sigma_c^2} = \gamma[1 + \lambda(s_t)].$$

- Since $\operatorname{Var}_t(r_i) = b_i^2 \sigma_c^2 + s_i^2$,

$$\boxed{\mathbb{E}_t[r_i] - r_{f,t} \approx b_i \Lambda_t \sigma_c^2 - \tfrac{1}{2}(b_i^2 \sigma_c^2 + s_i^2).}$$

- Covariance compensation rises when surplus consumption is low; the last term is the Jensen adjustment.

Calibration Logic and Countercyclical Premia

- The consumption process disciplines mean growth g and volatility σ_c. Given (γ, ϕ), choose β to match the average safe rate:
$$r_f = -\log \beta + \gamma g - \tfrac{1}{2}\gamma(1-\phi).$$
- The persistence parameter ϕ governs how slowly surplus consumption s_t returns toward $\bar{s}$ and therefore how persistent movements in risk premia, expected returns, and valuation ratios can be.
- The curvature parameter γ, together with (ϕ, σ_c), determines the steady-state surplus ratio
$$\bar{S} = \sigma_c \sqrt{\frac{\gamma}{1-\phi}},$$
and the level of effective risk aversion γ / S_t.
- The key pricing mechanism is state dependence:
$$\begin{aligned} \text{Good times: } S_t \uparrow \quad &\Rightarrow \quad \lambda(s_t) \downarrow \Rightarrow \sigma_t(m) \downarrow \Rightarrow \text{lower required premia,} \\ \text{Bad times: } S_t \downarrow \quad &\Rightarrow \quad \lambda(s_t) \uparrow \Rightarrow \sigma_t(m) \uparrow \Rightarrow \text{higher required premia.} \end{aligned}$$
- Higher required premia in bad times imply lower asset prices and generate predictable, countercyclical expected returns while the canonical specification keeps the risk-free rate constant.

$\hookrightarrow$ Calibration must therefore be judged jointly against the safe rate, equity premium, Sharpe ratios, return predictability, valuation-ratio volatility, and consumption–return comovement.

What Have We Learned?

- One pricing restriction ($1 = \mathbb{E}_t[M_{t+1} R^g_{i,t+1}]$) but four models: they differ in **what moves** M

Model	What changes?	Pricing mechanism	Main discipline / limitation
CRRA	Consumption growth drives the SDF	$m = \log \beta - \gamma \Delta c$: constant price per unit of consumption risk	Smooth consumption struggles jointly with premia, the safe rate, and price volatility
Simple habit	Stronger consumption-growth loading	$\gamma \to \gamma + \kappa$: consumption shocks receive greater SDF weight	Loading remains constant across states; the predetermined habit term also moves the safe rate
Rare disasters	Add low-probability, severe left-tail consumption risk	Disaster states receive very high marginal-utility weight	i.i.d. disasters can change average premia, but not by themselves generate persistent time variation
Campbell–Cochrane	Surplus consumption becomes a state variable	$\gamma[1+\lambda(s_t)]$ makes the price of ordinary consumption risk countercyclical	Risk premia and SDF volatility vary while the canonical specification keeps the safe rate constant

↪ Disasters primarily change the **distribution of bad-state risk**; Campbell–Cochrane changes its **state-dependent price**.

↪ In every model, the same SDF must price both safe and risky assets—the central discipline of consumption-based asset pricing.

▷ Next session: Dynamic Asset Pricing.

Thanks for your attention! See you next time!

Bibliography I

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