# Asset Pricing Theory: S01 Fundamentals of Asset Pricing

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# Structure of the course

- Email: amedeo.andriollo@dauphine.psl.eu
- Grading follows a 30-70 rule: 70% final exam, 30% project/homework (3rd and 5th sessions).
- Office hours: feel "free" to (DO) come.
- Slides/materials are updated regularly. Report a typo/mistake: corrections help the class. Slides are partially based on last year's (prof. Imbet).
- This session:
  - No-arbitrage and the Law of One Price; complete markets and state prices; Stochastic Discount Factor; SDF and expected returns.
  - Main reference: Campbell (2017)'s Chp.4 (sections: 4.1.1–3; 4.2.1-2; 4.3.1).

▷ Next session: consumption-based asset pricing.

# What is an Asset?

- An asset is a **resource** that is **owned** by an individual or entity and is expected to provide future economic benefits or costs.
- Assets can take various forms, including physical assets (e.g., real estate, machinery) and intangible assets (e.g., patents, trademarks).
- Financial assets, such as stocks and bonds, represent claims on future cash flows linked to the underlying assets or the issuing entity.
- **Asset Pricing** is the process of determining the **fair** value of an asset, taking into account its expected future cash flows, risks, and the time value of money.
- The concept of **fairness** is delicate, and hopefully by the end of this course we should understand the difference between what prices **should** be and what they actually are.

# State of Nature

- The future is uncertain, and since most asset holders (investors) or asset suppliers (e.g. firms) are risk-averse, uncertainty places an important role in determining prices.
- A **state of nature** is a possible outcome of the world at a given point in time.
- States of nature are clearly **continuous** and therefore **infinitely uncountable**.
- Discretizing states of nature is normally a simplification of reality that can provide a more manageable framework.
- We will start with a discrete version of our analysis and later provide continous counterparts.
- Imagine a world with discrete time $t \in \{0, 1, 2, \ldots\}$ and a finite set of states of nature $S = \{s_1, s_2, \ldots, s_n\}$.
- A payoff of an asset is a $|S|$-dimensional vector $X = (X_1, \ldots, X_n)$, where $X_i$ is the payoff in state $s_i$.
- For now consider the case in which known probabilities are assigned to each state of nature denoted by $\pi(s_i)$.

# Conceptual Route: From Payoffs to Prices

- Having defined assets, payoffs and their associated probabilities, how to study the economy?
  $\hookrightarrow$ Set a utility function, consumption process, production technology, supply-demand equilibrium...
- This session starts before by asking three sequential questions:
  1. what can be learned from the implication of the financial markets not offering arbitrage opportunities (next slide):
  $$\text{Payoffs} \Longrightarrow \text{No Arbitrage} \Longrightarrow \text{State Prices} \Longrightarrow \text{SDF}.$$
  2. when the available assets span every possible state-contingent payoff:
  $$\text{Completeness} \Longrightarrow \text{Uniqueness of State Prices/SDF}.$$
  3. what is the connection between prices to expected returns:
  $$1 = \mathbb{E}[MR^g] \quad \Longrightarrow \quad \text{risk premia depend on covariance with } M.$$

$\hookrightarrow$ Next session will provide **economic structure** for $M$ using preferences and consumption.

# Arbitrage

- There is a widely used expression in finance: **"There is no such thing as a free lunch."**
- In other words, if something appears to be too good to be true, it probably is.
- **Arbitrage** refers to the practice of taking advantage of price differences in different markets or forms.
- In an efficient market, arbitrage opportunities are quickly exploited and eliminated.
- Early work by Fama during the 1960s laid the foundation for modern asset pricing theory and the efficient market hypothesis.

# The Efficient Market Hypothesis

- The view about efficiency in financial markets accepted in the 60s/70s states that assets quickly react to new information.
    - ▷ Good news (higher future cashflows) tend to **increase asset prices**, while bad news tend to **decrease asset prices**.
    - ▷ Modern versions of this hypothesis incorporate limits to arbitrage, private information, and behavioral biases.

In a nutshell, there are three forms of market efficiency:

- **Weak form**: Prices reflect all past market data (e.g., prices, volumes).
- **Semi-strong form**: Prices reflect all publicly available information (e.g., financial statements, news).
- **Strong form**: Prices reflect all information, both public and private (e.g., insider information).

# Arbitrage and the EMH (1/2)

- A portfolio is a **linear** combination of assets: given $N$ assets with payoffs $X^{(1)}, \ldots, X^{(N)}$ where each $X^{(i)}$ is a vector, a portfolio is defined by weights $\theta = (\theta_1, \theta_2, \ldots, \theta_N)$.
- The expected payoff of the portfolio in each state of nature is given by the weighted sum of the payoffs of the individual assets:

$$X^P(s) = \sum_{i=1}^{N} \theta_i X^{(i)}(s)$$

$\hookrightarrow$ For **payoffs**, the weights do not need to sum to one, but they will sum to one for returns.

- An arbitrage opportunity is a portfolio with zero or negative price, but non-negative expected payoffs in all states of nature (and positive in at least one state): a risk-free profit opportunity that arises when the mkt misprices a portfolio of assets.

# Arbitrage and the EMH (2/2)

- Denote by $P$ the price of the portfolio, given by the weighted sum of the prices of the individual assets (e.g. buy 2 shares of TSLA and 2 of AAPL)

$$P = \sum_{i=1}^{N} \theta_i P^{(i)}$$

- A formal definition of an arbitrage states that there exist weights $\theta = (\theta_1, \theta_2, \dots, \theta_N)$ such that:
  1. The expected payoff of the portfolio is non-negative in all states (and positive in at least one):

$$X^P(s) = \sum_{i=1}^{N} \theta_i X^{(i)}(s) \geq 0 \quad \forall s \in S$$

  2. The price of the portfolio is zero or negative:

$$P = \sum_{i=1}^{N} \theta_i P^{(i)} \leq 0$$

  3. Strictly positive payoff with positive probability for at least one state:

$$\mathbb{P}(X^P(s) > 0) > 0$$

# Existence of an Arbitrage Opportunity

- An arbitrage strategy is the solution to the following system of three equations:

$$X^P(s) = A\theta \geq 0 \tag{1}$$

$$P = \vec{p}^T\theta \leq 0 \tag{2}$$

$$1^T A\theta \geq \epsilon > 0 \tag{3}$$

where $A$ is the matrix of payoffs, $\vec{p}$ is the vector of prices, $1^T$ is a vector of ones, and $\epsilon$ is a small positive number. This deals with the strict inequality imposed by condition 3.

# Example

- Let us consider the following setup:
  - ▷ 2 assets (a bond and a stock) and two states of nature: good and bad.
  - ▷ Bond has a price of $1$ and pays $1$ in both states.
  - ▷ Stock has a price of $1.5$, pays $1$ in state good and $0$ in state bad.
- Arbitrage portfolio: $\theta = (1, -1)$.
- Current cost: $P = 1 - 1.5 = -0.5$. State payoffs: $X^P = (1 - 1, 1 - 0) = (0, 1)$.
- **Mkt Dynamics and Equilibrium:**
  - Buying one bond and short selling one stock gives a risk-free profit of $0.5$...
  - ...but the, what stops the investor from buying 2 bonds and short selling 2 stocks?..
  - ..Or better, to sell all his assets, ask for an excessive amount of loans, buy millions of bonds and short millions of stocks?

  $\hookrightarrow$ An arbitrage is a deviation from an equilibrium, and excessive demand and supply will quickly adjust prices.

# Complete Markets

- A market is **complete** if any payoff vector $X \in \mathbb{R}^{|S|}$ can be replicated by a portfolio of **traded** assets.
- Formally: the span of traded payoffs $\{X^{(1)}, \ldots, X^{(N)}\}$ is the whole space $\mathbb{R}^{|S|}$.
- **Implication:**
  For any payoff $X = (X(s_1), \ldots, X(s_{|S|}))$, there exist weights $\theta = (\theta_1, \ldots, \theta_N)$ such that:

$$X(s) = \sum_{i=1}^{N} \theta_i X^{(i)}(s), \quad \forall s \in S.$$

# Law of One Price and Stochastic Discount Factor

- Consider two portfolios $A$ and $B$ with payoffs $X^A$ and $X^B$.
- Under no arbitrage, if $\forall s, X^A(s) = X^B(s)$ (same payoffs for all states), then: $P^A = P^B$.
  $\hookrightarrow$ If not one price, a quick arbitrage strategy of buy-cheap & sell-expensive can be formed.
- Given the probabilities $\pi(s)$ of the states, is there a **map** between payoffs to prices?
- The Stochastic Discount Factor is a positive random variable $M$ such that the price of any asset is given by the discounted expected payoff:

$$P = \mathbb{E}[M(s)X(s)] = \sum_{s \in S} \pi(s)M(s)X(s)$$

- Under no arbitrage opportunities, there exists a strictly positive Stochastic Discount Factor (SDF).
- (Objective) probabilities are not risk-neutral: states do not discount the risk.

# Arrow-Debreu Securities

- An **Arrow–Debreu security**, $\delta^{(s)}$, pays $1$ in state $s$ and $0$ in all other states:

$$\delta^{(s)}(s') = \begin{cases} 1 & \text{if } s' = s, \\ 0 & \text{otherwise.} \end{cases}$$

- If markets are complete, each $\delta^{(s)}$ can be replicated with some portfolio $\theta^{(s)}$.

- Existence of Arrow-Debreu Securities:
  - ▷ Stack payoffs in the matrix $A \in \mathbb{R}^{|S| \times N}$ with entries $[A]_{si} = X^{(i)}(s)$.
  - ▷ **Completeness** $\iff$ $\text{rank}(A) = |S|$ (i.e., full row rank).
  - ▷ For each state $s$, there exists a portfolio $\theta^{(s)}$ such that

$$A\,\theta^{(s)} = \delta^{(s)}.$$

  - The price of $\delta^{(s)}$ is then called the **state price**:

$$q(s) = \sum_{i=1}^{N} \theta_i^{(s)}\, p^{(i)}.$$

# Pricing Any Payoff with State Prices

- Any payoff $X \in \mathbb{R}^{|S|}$ can be decomposed as $X = \sum_{s\in S} X(s)\,\delta^{(s)}$.
- By linearity of pricing:

$$P(X) = \sum_{s\in S} X(s)\,q(s) = \sum_{s\in S} \pi(s)\,M(s)\,X(s), \quad M(s) = q(s)/\pi(s)$$

↪ In Arrow-Debreu framework, the price of any asset is weighted by $\{q(s)\}$.

- Define the risk-free asset as the asset which pays $1$ in all states. Its price is thus $\sum_{s\in S} q(s) = 1/(1+r)$, with $r$ beings the risk-free rate.
- Accordingly, define the **risk-neutral** probabilities:

$$\pi^{\mathbb{Q}}(s) = \frac{q(s)}{\sum_{s'\in S} q(s)} = q(s)(1+r)$$

so the price can be written as:

$$P(X) = \sum_{s\in S} X(s)\,q(s) = \sum_{s\in S} X(s)\,\frac{\pi^{\mathbb{Q}}(s)}{1+r} = \frac{1}{1+r}\sum_{s\in S} X(s)\,\pi^{\mathbb{Q}}(s) = \frac{1}{1+r}\mathbb{E}^{\mathbb{Q}}[X]$$

↪ A **change of measure** from physical probabilities $\pi$ to risk-neutral probabilities $\pi^{\mathbb{Q}}$.

# Fundamental Theorem of Asset Pricing (1/2)

- We now return to the finite-state economy. Let $A \in \mathbb{R}^{|S| \times N}$ denote the payoff matrix and $p \in \mathbb{R}^N$ the vector of date-0 prices.
- A portfolio $\theta \in \mathbb{R}^N$ generates the complete date-0/date-1 net trade:

$$z(\theta) = \begin{pmatrix} -p^\top \theta \\ A\theta \end{pmatrix} \in \mathbb{R}^{1+|S|}.$$

- Define the set of attainable net trades:

$$\mathcal{Z} = \left\{ \begin{pmatrix} -p^\top \theta \\ A\theta \end{pmatrix} : \theta \in \mathbb{R}^N \right\}.$$

- **No Arbitrage** is equivalent to:

$$\mathcal{Z} \cap \mathbb{R}_+^{1+|S|} = \{0\}.$$

↪ There is no feasible portfolio delivering a non-negative cash flow at every date/state and a strictly positive cash flow somewhere.

*Reference: (Campbell, 2017, Sec. 4.2.2, p. 92).*

# Fundamental Theorem of Asset Pricing (2/2)

- Separation argument: $\mathcal{Z}$ is a linear subspace and $\mathbb{R}_+^{1+|S|}$ is the positive cone.
- Under NA, the Separating Hyperplane Theorem implies the existence of a strictly positive vector

$$\widehat{q} = \begin{pmatrix} q_0 \\ q \end{pmatrix} \gg 0$$

  that separates attainable net trades from strictly positive payoffs.
- Because $\mathcal{Z}$ is a **linear subspace**, the separating functional must be zero on every $z(\theta) \in \mathcal{Z}$:

$$-q_0 p^\top \theta + q^\top A\theta = 0 \qquad \forall\theta.$$

- Normalizing $q_0 = 1$:

$$p = A^\top q, \qquad q \gg 0.$$

$\hookrightarrow$ **Fundamental Theorem of Asset Pricing:**

$$\boxed{\text{NA} \quad \Longleftrightarrow \quad \exists\, q \gg 0 \text{ such that } p = A^\top q.}$$

- $q(s)$ is the **state price**: the date-0 price of one unit of consumption delivered only in state $s$.

*Reference: (Campbell, 2017, Sec. 4.2.2, pp. 92–93).*

# Existence of the SDF

- Suppose a one-period economy with $n$ assets. Let $\mathcal{X}$ be the linear space of payoffs at $1$ (e.g. bounded random variables $L^\infty(\Omega, \mathcal{F}, \mathbb{P})$). Each asset i has prices $p_i$ and payoffs $X_i \in \mathcal{X}$.
  The pricing map on portfolios is linear:

$$f\left(\sum_i \theta_i X_i\right) = \sum_i \theta_i p_i$$

$\hookrightarrow$ **No arbitrage (NA):** any nonnegative payoff must have a nonnegative price.
If the payoff is strictly positive in some state, its price must be strictly positive.

- Define the **cone of zero-cost or negative payoffs**:

$$\mathcal{C} = \left\{ Y = \sum_i \theta_i X_i : \sum_i \theta_i p_i \le 0 \right\}.$$

This is the set of payoffs that can be obtained at zero or negative cost.

# Step 1: Hyperplane Separation Theorem

- ECONOMIC IDEA:
  Under NA, the set of non positive costs $\mathcal{C}$ cannot produce strictly positive payoffs. That is, $\mathcal{C}$ intersects the positive cone of payoffs $\mathcal{X}_+$ only at the origin: $\mathcal{C} \cap \mathcal{X}_+ = \{0\}$.
- MATHEMATICAL IDEA:
  Both $\mathcal{C}$ and $\mathcal{X}_+$ are convex. If two convex sets intersect only at $0$, the **Separating Hyperplane Theorem** guarantees the existence of a linear functional $\varphi$ that "separates" them.
- RESULT:
  There exists a nonzero linear functional $\varphi : \mathcal{X} \to \mathbb{R}$ such that:

$$\varphi(Y) \leq 0 \quad \forall Y \in \mathcal{C}, \qquad X \geq 0 \implies \varphi(X) \geq 0.$$

- INTERPRETATION:
  $\varphi$ is a *positive pricing rule*: it prices every portfolio consistently with NA and vanishes on zero-cost trades. Geometrically, $\varphi$ is the "supporting hyperplane" separating feasible trades from arbitrage opportunities.

# Step 2: Representation Theorem

- REPRESENTATION: positive linear functionals on payoff spaces can be represented as expectations.
  $\hookrightarrow$ **Theorem (Riesz / Yosida-Hewitt)**: if $\varphi : \mathcal{X} \to \mathbb{R}$ is positive and continuous on $L^p$ (e.g. $L^2$), then **there exists** a nonnegative random variable $M$ such that: $\varphi(X) = \mathbb{E}[MX]$, $\forall X \in \mathcal{X}$.
- SIMPLE PROOF IN FINITE STATES: Let $S = \{s_1, \ldots, s_n\}$ with associated probabilities $\pi(s_i) > 0$.
  1. Write $\psi(X) = a_1 X_1 + \cdots + a_n X_n$.
  2. Multiply and divide by $\pi_i$:

$$\psi(X) = \sum_{i=1}^{n} \pi_i \frac{a_i}{\pi_i} X_i.$$

  3. Define $M_i = a_i / \pi_i$. Hence:

$$\psi(X) = \sum_{i=1}^{n} \pi_i M_i X_i = \mathbb{E}[MX].$$

$\hookrightarrow$ $\psi(\cdot)$ is an expectation w.r.t. the random variable $M$.
If $\psi(\cdot)$ is strictly positive for nonnegative $X \neq 0$, then all $a_i > 0$ and hence $M_i > 0$.

# Prices as Expectations

- From Non-Arbitrage (NA) to the Stochastic Discount Factor (SDF) in two steps:
  - ▷ $1\#$ Step: NA $\implies \mathcal{C} \cap \mathcal{X}_+ = \{0\}$. By the hyperplane theorem, there exists $\varphi$ with:

$$\forall Y \in \mathcal{C}, \varphi(Y) \leq 0, \quad X \geq 0 \implies \varphi(X) \geq 0.$$

  - ▷ $2\#$ Step: By the representation theorem, $\varphi(X) = \mathbb{E}[MX]$, with $M$ the SDF. Recall that $\varphi$ is strictly positive for NA payoffs, meaning that: $M > 0$ a.s.

↪ Prices reads as expectations.

- For any portfolio $\theta$, $f\left(\sum_i \theta_i X^{(i)}\right) = \sum_i \theta_i p^{(i)}$.
  In particular, for $f = \varphi$:

$$\sum_i \theta_i p^{(i)} = \mathbb{E}\left[M \sum_i \theta_i X^{(i)}\right].$$

↪ for each asset $i = 1, \ldots, N$: $p^{(i)} = \mathbb{E}[MX^{(i)}]$.

# Uniqueness of the SDF and Extensions

- Uniqueness:
  - Suppose $M$ and $M'$ price all traded assets: $p^{(i)} = \mathbb{E}[MX^{(i)}] = \mathbb{E}[M'X^{(i)}], \quad i = 1, \dots, N.$
  - In state-price form, we have then: $A^\top q = p$ and $A^\top q' = p$, where for each state $s$:

$$q(s) = \pi(s)\, M(s), \qquad q'(s) = \pi(s)\, M'(s).$$

  - Subtracting $A^\top (q - q') = 0$.
  - By completeness of the markets: $\ker(A^\top) = \{0\} \Rightarrow q = q'$.
  - With $\pi(s) > 0$ for all $s$, it follows that $M(s) = M'(s)$.

- Extension to Continuous States:
  - What do we need to extend these concepts to infinite dimensional spaces?
    - ✓ Generalized separating hyperplane theorems (exist);
    - ✓ Generalized representation theorems (exist);
    - ☐ Uniqueness is a challenge, since infinite dimensional payoff spaces can only be spanned **approximately**.

# Projection Geometry of the SDF (1/2)

- Let us consider the discrete case first.
  Define inner product:

$$\langle X, Y\rangle = \sum_{s\in S} \pi(s)X(s)Y(s).$$

- The span of traded payoffs:

$$\mathcal{S} = \{A\theta : \theta \in \mathbb{R}^N\}.$$

- Any SDF $M$ decomposes uniquely as:

$$M = M_\parallel + M_\perp, \quad M_\parallel \in \mathcal{S},\ M_\perp \in \mathcal{S}^\perp.$$

- For any traded payoff $Y \in \mathcal{S}$,

$$\mathbb{E}[MY] = \mathbb{E}[M_\parallel Y].$$

# Projection Geometry of the SDF (2/2)

- **The projection decomposition:**
  Let us write $M = M_{\|} + M_{\perp}$, with: i) $M_{\|} = \text{proj}_{\mathcal{S}}(M)$ being the orthogonal projection of $M$ onto $\mathcal{S}$, and ii) $M_{\perp} \in \mathcal{S}^{\perp}$ being the orthogonal component such that $\langle M_{\perp}, Y\rangle = 0$ for any $Y \in \mathcal{S}$.
- **Pricing Implications:**
  For any traded payoff $Y \in \mathcal{S}$:

$$\langle M, Y\rangle = \langle M_{\|} + M_{\perp}, Y\rangle = \langle M_{\|}, Y\rangle + \underbrace{\langle M_{\perp}, Y\rangle}_{=0} = \langle M_{\|}, Y\rangle$$

↪ All SDFs with the same projection $M_{\|}$ yield identical prices:

$$M_1 = M_{\|} + M_{\perp}^{(1)}, \quad M_2 = M_{\|} + M_{\perp}^{(2)} \quad \Rightarrow \quad \langle M_1, Y\rangle = \langle M_2, Y\rangle \text{ for all } Y \in \mathcal{S}$$

- **Key Insight:**
  Only the projection of $M$ onto $mathcalS$ matters for pricing traded assets.

Thanks for your attention! See you next time!

# Bibliography I

Campbell, J. Y. (2017). *Financial decisions and markets: a course in asset pricing*. Princeton University Press.