Asset Pricing Theory

In-Class Exam 1 – Solutions

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Part A, Correct answers: A, D, A, B, B, D, C.

Q1.A.

Let q_s be the date-0 price of one unit delivered only in state s. With unrestricted trading and positive physical probabilities, no arbitrage requires strictly positive state prices. The bond and stock require

$$q_1 + q_2 + q_3 = 1, \qquad 2q_1 + q_2 = 1.$$

Subtract the former to the latter:

$$(2q_1 + q_2) - (q_1 + q_2 + q_3) = 0 \quad \Longrightarrow \quad q_1 - q_3 = 0.$$

The price of C imposes $q_1 + q_3 = p$. Therefore

$$q_1 = q_3 = p/2, \qquad q_2 = 1 - p, \qquad z = q_1 + q_2 = 1 - p/2.$$

All three state prices are positive exactly when $0 < p < 1$. For those prices every nonzero nonnegative payoff has positive cost, thus ruling out arbitrage.

Completeness is about independent payoffs: If $aB + bS + cC = 0$, the second-state equation is $a + b = 0$, the third is $a + c = 0$, and the first is $a + 2b + c = 0$. Substitution gives $-2a = 0$, so $a = b = c = 0$. The three claims span all three states. The fourth claim is redundant.

Q2.D.

The SDF represents the Arrow Debreu price as an expectation under a specified physical measure. For the indicator payoff of state s,

$$q_s = \mathbb{E}[M\mathbf{1}_{\{s\}}] = \pi_s M_s.$$

Consequently the representation must change when the positive physical probabilities change while the Arrow Debreu prices do not change:

$$M^{\text{old}} = \left(\frac{1/4}{1/2}, \frac{1/2}{1/2}\right) = (1/2, 1),$$
$$M^{\text{new}} = \left(\frac{1/4}{1/4}, \frac{1/2}{3/4}\right) = \left(1, \frac{1}{2}\frac{4}{3}\right) = (1, 2/3).$$

For every payoff vector X, both expectations give the same price. The price of one unit in every state is $P_0 = \sum_s q_s = 3/4$ (its gross risk-free return is $R^g_{f,0} = 4/3$). The probabilities used for risk-neutral valuation are normalized state prices:

$$\pi^Q_s = \frac{q_s}{P_0}, \qquad \pi^Q = (1/3, 2/3), \qquad \text{price}(X) = P_0 \mathbb{E}^Q[X].$$

Q3.A.

Zero consumption exposure removes the risk premium but not the Jensen adjustment between log and gross returns. The pricing restriction for the risk and risk-free assets is:

$$1 = \mathbb{E}_t[M_{t+1}R^g_{t+1}], \qquad 1 = R^g_{f,t}\mathbb{E}_t[M_{t+1}].$$

Joint normality and zero conditional covariance imply conditional independence of Δc_{t+1} and r_{t+1}. Taking exponentials and multiplying by constants preserves independence: M_{t+1} and R^g_{t+1} are conditionally independent as well. Therefore

$$1 = \mathbb{E}_t[M_{t+1}]\mathbb{E}_t[R^g_{t+1}], \qquad \mathbb{E}_t[R^g_{t+1}] = \frac{1}{\mathbb{E}_t[M_{t+1}]} = R^g_{f,t}.$$

Conditional normality of r_{t+1} also gives the exact identity

$$\begin{aligned} \mathbb{E}_t[R^g_{t+1}] &= \exp\{\mathbb{E}_t[r_{t+1}] + v_t/2\}, \\ r_{f,t} = \log \mathbb{E}_t[R^g_{t+1}] &= \mathbb{E}_t[r_{t+1}] + v_t/2. \end{aligned}$$

Since $v_t > 0$, expected log excess return is strictly negative.

Q4.B.

The value derivatives are

$$V_W = AW^{-\gamma} > 0, \qquad V_{WW} = -\gamma A W^{-\gamma-1} < 0.$$

Consumption FOC:

$$U'(C) - V_W = C^{-\gamma} - AW^{-\gamma} = 0 \quad \Longrightarrow \quad C^* = A^{-1/\gamma}W.$$

For the risky share,

$$\begin{aligned} 0 &= V_W W(\mu - r) + V_{WW}\omega W^2\sigma^2 = AW^{1-\gamma}\{(\mu - r) - \gamma\omega\sigma^2\}, \\ \Longrightarrow\ \omega^* &= \frac{\mu - r}{\gamma\sigma^2}. \end{aligned}$$

Holding A and investment opportunities fixed, the share stays constant but the monetary position $\omega^* W$ doubles with wealth. Consumption doubles as well. Homogeneity concerns proportions of wealth, not constant consumption levels or constant dollar positions.

Q5.B.

Let $q_{i,t}$ be the marginal cost/value of installed capital. Optimal investment gives

$$q_{i,t} = 1 + G_I(I_{i,t}, K_{i,t}) = 1 + \kappa(x_i - \delta) = \mathbb{E}_t[M_{t+1}V^i_{K,t+1}].$$

Here one additional unit of $I_{i,t}$ costs $1+G_I$ now and adds one unit to $K_{i,t+1}$, with marginal continuation payoff $V^i_{K,t+1}$. Define $H = \mathbb{E}_t[V^i_{K,t+1}] > 0$ and $M = \mathbb{E}_t[M_{t+1}] > 0$, and $k_i = \mathrm{Cov}_t(M_{t+1}, V^i_{K,t+1})$. From the Euler condition on capital:

$$q_{i,t} = MH + k_i.$$

Since $k_A < k_B$, $q_{A,t} < q_{B,t}$. Inverting the investment FOC,

$$x_i = \delta + \frac{q_{i,t} - 1}{\kappa} \qquad \Longrightarrow \; x_A < x_B.$$

The physical gross return is payoff per unit marginal cost, $R^{I,g}_{i,t+1} = V^i_{K,t+1}/q_{i,t}$. Consequently

$$\mathbb{E}_t[R^{I,g}_{A,t+1}] = \frac{H}{q_{A,t}} > \frac{H}{q_{B,t}} = \mathbb{E}_t[R^{I,g}_{B,t+1}].$$

Q6.D.

Marginal q_t is valuing capital installed for next period VS. $V_{K,t}$ is valuing capital inherited today. Ex-dividend value:

$$P_t = V_t(K_t) - D_t = \mathbb{E}_t[M_{t+1}V_{t+1}(K_{t+1})].$$

For investment, the FOC:

$$1 + G_{I,t} = \mathbb{E}_t[M_{t+1}V_{K,t+1}(K_{t+1})] = q_t.$$

Due homogeneity (at the optimum), differentiating $V_{t+1}(\lambda K) = \lambda V_{t+1}(K)$ with respect to λ gives

$$V_{K,t+1}(\lambda K)K = V_{t+1}(K).$$

Set $\lambda = 1$ and divide by K: $V_{K,t+1}(K) = V_{t+1}(K)/K$. Substituting into the investment FOC yields

$$q_t = \mathbb{E}_t\left[M_{t+1}\frac{V_{t+1}(K_{t+1})}{K_{t+1}}\right] = \frac{\mathbb{E}_t[M_{t+1}V_{t+1}(K_{t+1})]}{K_{t+1}} = \frac{P_t}{K_{t+1}} \equiv Q_t.$$

The denominator can leave the conditional expectation because investment (and so K_{t+1}), is chosen with date-t information.

Q7.C.

One extra consumption unit of J_t creates μ_t capital units. With no adjustment costs:

$$-1 + \mu_t q_t = 0, \qquad q_t = 1/\mu_t.$$

The envelope condition is $V_{K,t+1} = F_{K,t+1} + (1-\delta)q_{t+1}$. Divide the marginal payoff by today's marginal cost:

$$R^{I,g}_{t+1} = \frac{F_{K,t+1} + (1-\delta)/\mu_{t+1}}{1/\mu_t} = \mu_t\left[F_{K,t+1} + \frac{1-\delta}{\mu_{t+1}}\right].$$

Capital optimality $q_t = \mathbb{E}_t[M_{t+1}V_{K,t+1}]$ implies $1 = \mathbb{E}_t[M_{t+1}R^{I,g}_{t+1}]$. Taking conditional expectations of the returns:

$$\mathbb{E}_t[R^{I,g}_{t+1}] = \mu_t\{\mathbb{E}_t[F_{K,t+1}] + (1-\delta)\mathbb{E}_t[1/\mu_{t+1}]\}.$$

A higher future efficiency, state by state, lowers the replacement value: $\partial R^{I,g}_{t+1}/\partial\mu_{t+1} = -\mu_t(1-\delta)/\mu^2_{t+1} < 0$ (holding the other terms fixed). A rise in the mean alone does not order the expected reciprocal.

Part B - Q8: Income risk and precautionary saving

Q8(a). Budgets and the consumption Euler equation [4 points].

The household chooses how much consumption to transfer from date 0 to date 1:

$$c_0 + a_0 = w, \qquad c_1 = R^g_1 a_0 + Y_1.$$

The saving decision a_0 is made before the realization of (R^g_1, Y_1). Substituting $c_0 = w - a_0$ and $c_1 = R^g_1 a_0 + Y_1$ into expected utility, the household solves

$$\max_{0\le a_0<w} \{\log(w-a_0) + \beta\mathbb{E}_0[\log(R^g_1 a_0 + Y_1)]\}.$$

For an interior choice $a_0 > 0$, FOC:

$$-\frac{1}{w-a_0} + \beta\mathbb{E}_0\left[\frac{R^g_1}{R^g_1 a_0 + Y_1}\right] = 0.$$

Using the two budgets, this becomes the Euler equation:

$$\boxed{\frac{1}{c_0} = \beta\mathbb{E}_0\left[\frac{R^g_1}{c_1}\right].}$$

Interpretation of the Euler: Saving one additional unit reduces current consumption by one unit, with marginal utility cost $u'(c_0) = 1/c_0$. At date 1, the same unit of saving produces R^g_1 units of consumption, each valued at marginal utility $u'(c_1) = 1/c_1$. The Euler equation equates the current marginal cost with the discounted expected marginal benefit of saving.

Q8(b). Deterministic benchmark [5 points].

When $R^g_1 = \bar{R}$ and $Y_1 = \bar{Y}$, future consumption is

$$c_1 = \bar{R}a_0 + \bar{Y}.$$

The Euler equation becomes

$$\frac{1}{w-a_0} = \frac{\beta\bar{R}}{\bar{R}a_0 + \bar{Y}}.$$

Consequently, expressing a_0:

$$\boxed{a_0^{\text{det}} = \frac{\beta \bar{R} w - \bar{Y}}{(1+\beta)\bar{R}} = \frac{\beta w - \bar{Y}/\bar{R}}{1+\beta}.}$$

Our assumed inequality $\beta \bar{R} w > \bar{Y}$ ensures $a_0^{\text{det}} > 0$.

Explanation: Higher β increases the weight assigned to future consumption and therefore raises saving. Higher future income $\bar{Y}$ reduces the need to transfer resources from today to tomorrow, lowering saving. A higher gross return $\bar{R}$ makes saving more attractive in this log-utility benchmark with positive future income, and therefore raises optimal saving.

Q8(c). Income uncertainty and precautionary saving [6 points].

Now $R_1^g = \bar{R}$, while

$$Y_1 = \bar{Y} e^{\sigma Z - \sigma^2/2}, \qquad Z \sim N(0,1).$$

For a standard normal random variable, we have: $\mathbb{E}[e^{uZ}] = e^{u^2/2}$. Therefore,

$$\mathbb{E}_0[Y_1] = \bar{Y} e^{-\sigma^2/2} \mathbb{E}_0[e^{\sigma Z}] = \bar{Y} e^{-\sigma^2/2} e^{\sigma^2/2} = \bar{Y}.$$

Thus uncertainty changes the distribution of future income without changing its expected value. The Euler equation is

$$\frac{1}{w - a_0} = \beta \bar{R} \mathbb{E}_0 \left[\frac{1}{\bar{R} a_0 + Y_1} \right].$$

Effect of income uncertainty. For any given saving $a_0 \geq 0$, the function

$$y \longmapsto \frac{1}{\bar{R} a_0 + y}$$

is strictly convex for $y > 0$. By Jensen's inequality, since $\sigma > 0$,

$$\mathbb{E}_0 \left[\frac{1}{\bar{R} a_0 + Y_1} \right] > \frac{1}{\bar{R} a_0 + \mathbb{E}_0[Y_1]} = \frac{1}{\bar{R} a_0 + \bar{Y}}.$$

At fixed saving level then the marginal benefits:

$$\underbrace{\beta \bar{R} \mathbb{E}_0 \left[\frac{1}{\bar{R} a_0 + Y_1} \right]}_{\text{under income risk}} > \underbrace{\frac{\beta \bar{R}}{\bar{R} a_0 + \bar{Y}}}_{\text{under no uncertainty}}.$$

Let us evaluate this comparison at the deterministic optimum a_0^{det}: i) at that saving level, the marginal benefit under certainty equals $1/(w - a_0^{\text{det}})$, while ii) under income risk, the marginal benefit is strictly greater. Thus, the household must increase saving: this raises $1/(w - a_0)$ and reduces the expected marginal benefit, therefore: $\boxed{a_0^{\text{income risk}} > a_0^{\text{det}}.}$

Economic mechanism: When future income is uncertain, low-income realizations are particularly costly because marginal utility is high when consumption is low. The benefit of saving in those states outweighs the reduced marginal benefit in high-income states. The household consequently saves more as a precaution against adverse income realizations.

Q8(d). Return uncertainty and optimal saving [7 points].

Now future income is certain, $Y_1 = \bar{Y}$, but the saving return is random:

$$R_1^g = \bar{R} e^{\sigma Z - \sigma^2/2}.$$

Using the same normal exponential-moment formula,

$$\mathbb{E}_0[R_1^g] = \bar{R} e^{-\sigma^2/2} \mathbb{E}_0[e^{\sigma Z}] = \bar{R}.$$

Expected returns are therefore unchanged relative to the deterministic benchmark, $\mathbb{E}_0[R_1^g] = \bar{R}$. However, the return on saving is now uncertain. Recalling the date-1 budget constraint,

$$c_1 = R_1^g a_0 + \bar{Y},$$

and substituting it into the general consumption Euler equation derived in part (a),

$$\frac{1}{c_0} = \beta \mathbb{E}_0 \left[\frac{R_1^g}{c_1} \right],$$

we obtain

$$\frac{1}{w - a_0} = \beta \mathbb{E}_0 \left[\frac{R_1^g}{R_1^g a_0 + \bar{Y}} \right].$$

Notice that uncertainty now affects both the marginal payoff from saving (the numerator R_1^g) and future consumption (the denominator $R_1^g a_0 + \bar{Y}$).

Effect of return uncertainty. Unlike income risk, the random return now appears in both the numerator and denominator. For any given $a_0 > 0$, we can write

$$\frac{R_1^g}{R_1^g a_0 + \bar{Y}} = \frac{1}{a_0} - \frac{\bar{Y}}{a_0} \frac{1}{R_1^g a_0 + \bar{Y}}.$$

Similar to before, because the function

$$R \longmapsto \frac{1}{R a_0 + \bar{Y}}$$

is strictly convex for positive R, Jensen's inequality gives

$$\mathbb{E}_0 \left[\frac{1}{R_1^g a_0 + \bar{Y}} \right] > \frac{1}{a_0 \mathbb{E}_0[R_1^g] + \bar{Y}} = \frac{1}{a_0 \bar{R} + \bar{Y}}.$$

However, multiplying this inequality by $-\bar{Y}/a_0 < 0$ reverses its direction. Adding $1/a_0$ to both sides yields

$$\mathbb{E}_0 \left[\frac{R_1^g}{R_1^g a_0 + \bar{Y}} \right] < \frac{1}{a_0} - \frac{\bar{Y}}{a_0(a_0 \bar{R} + \bar{Y})} = \frac{\bar{R}}{a_0 \bar{R} + \bar{Y}}.$$

Therefore, we have the opposite with respect to the previous scenario:

$$\underbrace{\beta \mathbb{E}_0 \left[\frac{R_1^g}{R_1^g a_0 + \bar{Y}} \right]}_{\text{under return risk}} < \underbrace{\frac{\beta \bar{R}}{a_0 \bar{R} + \bar{Y}}}_{\text{under no uncertainty}}.$$

Thus following the aforementioned logic (but reverse):

$$\boxed{0 < a_0^{\text{return risk}} < a_0^{\text{det}} < a_0^{\text{income risk}}.}$$

Please note that the assumed condition $\beta \bar{R} w > \bar{Y}$ ensures that saving remains strictly positive. Indeed, at $a_0 = 0$, the expected marginal benefit is $\beta \mathbb{E}_0[R_1^g]/\bar{Y} = \beta \bar{R}/\bar{Y} > 1/w$.

How to read it: When the saving return is high, each unit saved produces more future consumption, but marginal utility of that consumption is lower. Conversely, low returns coincide with higher marginal utility. Under logarithmic utility and positive future income, this interaction reduces the expected marginal benefit of saving relative to a certain return with the same mean.

Q8(e). Return uncertainty without future income [4 points].

Suppose $Y_1 = 0$, while $R_1^g > 0$ remains uncertain. The date-1 budget becomes

$$c_1 = R_1^g a_0.$$

The Euler equation is

$$\frac{1}{w - a_0} = \beta \mathbb{E}_0 \left[\frac{R_1^g}{R_1^g a_0} \right].$$

and since the gross return cancels inside the expectation we have:

$$\frac{1}{w - a_0} = \frac{\beta}{a_0}.$$

Finally:

$$\boxed{a_0^* = \frac{\beta}{1+\beta} w, \qquad c_0^* = \frac{1}{1+\beta} w.}$$

Quick explanation: With logarithmic utility and no future income, the return and its effect on future marginal utility cancel exactly in the Euler equation. Consequently, the optimal saving fraction does not depend on the distribution of the positive gross return.