Asset Pricing Theory: In-Class Exam 1

Duration: 65 minutes. Total: 50 points.

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Part A: Answer 6 out of 7 MCQs (no text needed). Part B: Solve the derivation.

Part A — Multiple-choice questions ($\sim 35'$)

Q1. Arbitrage [4 pts].

Three date-1 states have positive probabilities. A bond pays $B = (1, 1, 1)$ and a stock pays $S = (2, 1, 0)$, each cost 1 (i.e., same price). Two additional claims, paying $C = (1, 0, 1)$ and $D = (1, 1, 0)$, are offered at prices p and z. Which statement characterizes the enlarged market?

A. No arbitrage exactly when $0 < p < 1$ and $z = 1 - p/2$; the market is complete.
B. No arbitrage exactly when $0 \leq p \leq 1$ and $z = 1 - p/2$; the market is complete.
C. No arbitrage exactly when $0 < p < 1$ and $z = (1 + p)/2$; the market is incomplete.
D. No arbitrage exactly when $0 < p < 1$ and $z = 1 - p/2$; the market is incomplete.

Q2. Physical probabilities [4 pts].

In a complete two-state mkt, the Arrow-Debreu prices are $q = (1/4, 1/2)$. Under the physical probabilities $\pi^{\text{old}} = (1/2, 1/2)$ the SDF is $M^{\text{old}} = (1/2, 1)$. An analyst instead represents the same prices and payoffs using new probabilities $\pi^{\text{new}} = (1/4, 3/4)$. This is only a change of physical probabilities. Which pair gives the new SDF and the risk-neutral probabilities?

A. $M^{\text{new}} = (1/2, 1)$ and $\pi^{Q} = (1/7, 6/7)$.
B. $M^{\text{new}} = (1, 2/3)$ and $\pi^{Q} = (1/4, 3/4)$.
C. $M^{\text{new}} = (1/4, 3/2)$ and $\pi^{Q} = (1/3, 2/3)$.
D. $M^{\text{new}} = (1, 2/3)$ and $\pi^{Q} = (1/3, 2/3)$.

Q3. Log returns [4 pts].

Let $\Delta c_{t+1} = \log(C_{t+1}/C_t)$ and $M_{t+1} = \beta \exp(-\gamma \Delta c_{t+1})$, with $\beta, \gamma > 0$. Define $m_{t+1} = \log M_{t+1}$, $r_{t+1} = \log R^g_{t+1}$, and $r_{f,t} = \log R^g_{f,t}$, where the risk-free rate $R^g_{f,t}$ is known at t. Conditional on the past information $\mathcal{F}_t$, the pair $(\Delta c_{t+1}, r_{t+1})$ is jointly normal (notice that the variables are in log) with zero covariance and $v_t = \text{Var}_t(r_{t+1}) > 0$. Both returns satisfy the SDF pricing restriction. Which pair of conclusions is correct?

A. $\mathbb{E}_t[R^g_{t+1}]/R^g_{f,t} = 1$; $\quad \mathbb{E}_t[r_{t+1}] - r_{f,t} = -v_t/2$.
B. $\mathbb{E}_t[R^g_{t+1}]/R^g_{f,t} = e^{v_t/2}$; $\quad \mathbb{E}_t[r_{t+1}] - r_{f,t} = 0$.
C. $\mathbb{E}_t[R^g_{t+1}]/R^g_{f,t} = 1$; $\quad \mathbb{E}_t[r_{t+1}] - r_{f,t} = 0$.
D. $\mathbb{E}_t[R^g_{t+1}]/R^g_{f,t} = e^{-v_t/2}$; $\quad \mathbb{E}_t[r_{t+1}] - r_{f,t} = -v_t/2$.

Q4. Wealth and optimal policies [4 pts].

In the continuous-time consumption–portfolio problem (no labour income), $U(C) = C^{1-\gamma}/(1-\gamma)$, $\gamma > 0$, $\gamma \neq 1$. At a fixed date, $V(W) = AW^{1-\gamma}/(1-\gamma)$, $A > 0$, and the HJB tells us the interior optimum follows:

$$U(C) - V_W C + V_W \omega W(\mu - r) + \tfrac{1}{2} V_{WW} \omega^2 W^2 \sigma^2.$$

Here μ is the expected instantaneous risky return, r the safe rate, $\sigma > 0$ the risky volatility, and ω the risky wealth share. Choices are interior and $\mu - r > 0$. Holding A and the investment opportunities fixed, what happens when W doubles?

A. Consumption C^* is multiplied by $2^{1/\gamma}$; the risky holding $\omega^* W$ doubles.

B. Consumption C^* doubles; the risky holding $\omega^* W$ doubles.

C. Consumption C^* doubles; the risky holding $\omega^* W$ is multiplied by $2^{1/\gamma}$.

D. Consumption C^* doubles; the risky holding $\omega^* W$ is unchanged.

Q5. Investment and insurance [4 pts].

Two firms $i = A, B$ maximize (SDF-discounted) payouts with the same $M_{t+1} > 0$ facing the same function of adjustment costs: $G(I, K) = \frac{\kappa}{2}(I/K - \delta)^2 K$, $\kappa > 0$. Their positive marginal capital payoffs $V^i_{K,t+1}$ include marginal output, adjustment-cost effects and continuation value. Suppose

$$\mathbb{E}_t[V^A_{K,t+1}] = \mathbb{E}_t[V^B_{K,t+1}] > 0, \qquad \mathrm{Cov}_t(M_{t+1}, V^A_{K,t+1}) < \mathrm{Cov}_t(M_{t+1}, V^B_{K,t+1}) \leq 0.$$

Which pair of inequalities regarding investment rates $x_i = I_{i,t}/K_{i,t}$ and expected gross return on investment must hold?

A. $x_A < x_B$ and $\mathbb{E}_t[R^{I,g}_{A,t+1}] < \mathbb{E}_t[R^{I,g}_{B,t+1}]$.

B. $x_A < x_B$ and $\mathbb{E}_t[R^{I,g}_{A,t+1}] > \mathbb{E}_t[R^{I,g}_{B,t+1}]$.

C. $x_A > x_B$ and $\mathbb{E}_t[R^{I,g}_{A,t+1}] < \mathbb{E}_t[R^{I,g}_{B,t+1}]$.

D. $x_A = x_B$ and $\mathbb{E}_t[R^{I,g}_{A,t+1}] = \mathbb{E}_t[R^{I,g}_{B,t+1}]$.

Q6. Q and q. [4 pts].

A single-capital, all-equity firm chooses labour and investment to maximize

$$V_t(K_t) = \max_{L_t, I_t} \{D_t + \mathbb{E}_t[M_{t+1} V_{t+1}(K_{t+1})]\}, \qquad K_{t+1} = (1 - \delta)K_t + I_t.$$

Here $D_t = F(K_t, Z_t L_t) - w_t L_t - I_t - G(I_t, K_t)$, and $P_t = V_t(K_t) - D_t$ is ex-dividend value. Prices and the SDF are exogenous to the firm. Technology and feasible policies scale so that the differentiable *optimized* value satisfies $V_t(\lambda K) = \lambda V_t(K)$ in every state. For comparison, $h(L) = bL$ is degree-one homogeneous because $h(\lambda L) = \lambda h(L)$ $(b, \lambda > 0)$.

Investment is interior. Let q_t value one unit of capital installed for $t + 1$, in date-t consumption units; $K_t, K_{t+1} > 0$. Which identity is generally valid?

A. $q_t = P_t/K_t$.

B. $q_t = V_t(K_t)/K_t$.

C. $q_t = \mathbb{E}_t[V_{t+1}(K_{t+1})]/K_{t+1}$.

D. $q_t = P_t/K_{t+1}$.

Q7. Investment efficiency [4 pts].

With no adjustment costs, expenditure J_t (consumption units) installs $\mu_t J_t$ units of capital: $K_{t+1} = (1 - \delta)K_t + \mu_t J_t$, $0 < \delta < 1$. Efficiency $\mu_t > 0$ is observed at t. Along being at the optimal path, onsider changes in μ_{t+1}, holding μ_t and the marginal output $F_{K,t+1}$ fixed state by state (hint: recall the envelope condition regarding $V_{K,t+1}$). Which pair of statements is valid for general $\mu_t > 0$?

A. $\mathbb{E}_t[R^{I,g}_{t+1}] = \mu_t\{\mathbb{E}_t[F_{K,t+1}] + (1-\delta)\mathbb{E}_t[1/\mu_{t+1}]\}$; a higher $\mathbb{E}_t[\mu_{t+1}]$ necessarily lowers this return.

B. $\mathbb{E}_t[R^{I,g}_{t+1}] = \{\mathbb{E}_t[F_{K,t+1}] + (1-\delta)\mathbb{E}_t[1/\mu_{t+1}]\}/\mu_t$; a higher $\mathbb{E}_t[\mu_{t+1}]$ alone does not determine the change.

C. $\mathbb{E}_t[R^{I,g}_{t+1}] = \mu_t\{\mathbb{E}_t[F_{K,t+1}] + (1-\delta)\mathbb{E}_t[1/\mu_{t+1}]\}$; a higher $\mathbb{E}_t[\mu_{t+1}]$ alone does not determine the change.

D. $\mathbb{E}_t[R^{I,g}_{t+1}] = \mu_t\{\mathbb{E}_t[F_{K,t+1}] + (1-\delta)/\mathbb{E}_t[\mu_{t+1}]\}$; a higher $\mathbb{E}_t[\mu_{t+1}]$ necessarily lowers this return.

Part B - Derivation ($\sim 30'$)

Q8. Income Risk, Return Risk, and Optimal Saving [26 pts].

Consider a household that lives for two dates, $t = 0, 1$, and maximizes

$$\log c_0 + \beta \mathbb{E}_0[\log c_1], \qquad 0 < \beta \leq 1.$$

At date 0, the household has resources $w > 0$. It chooses consumption $c_0 > 0$ and saving $a_0 \geq 0$. Saving is invested in a single asset paying the gross return $R^g_1 > 0$ at date 1 (which may be uncertain). At date 1, the household receives labour income $Y_1 \geq 0$ and consumes all its resources. Borrowing is prohibited, and the labour income is taken as given and cannot be traded or insured. Throughout (see here below), $\bar{R} > 0$ and $\bar{Y} > 0$ are such that $\beta \bar{R} w > \bar{Y}$, thus ensuring positive optimal saving.
We examine three alternative environments (see points b-c-d), considering one source of uncertainty at a time.

(a) **[4 pts]** Write the two sequential budget constraints and substitute them into the household's objective. Derive the consumption Euler equation for an interior saving choice, allowing both R^g_1 and Y_1 to be random variables (but exogenous to the system). Briefly interpret its economic meaning.

(b) **[5 pts]** *No uncertainty.* Suppose $R^g_1 = \bar{R}$ and $Y_1 = \bar{Y}$. Derive optimal saving a_0^{det} explicitly. Explain briefly how it depends on β, $\bar{R}$, and $\bar{Y}$.

(c) **[6 pts]** *Risky income.* Suppose the saving return is certain/known, $R^g_1 = \bar{R}$, but future income is

$$Y_1 = \bar{Y} \exp\left(\sigma Z - \frac{\sigma^2}{2}\right), \qquad Z \sim N(0,1), \quad \sigma > 0.$$

Show that $\mathbb{E}_0[Y_1] = \bar{Y}$. Using the consumption Euler equation and Jensen's inequality, determine whether income uncertainty increases or decreases optimal saving relative to part (b). Explain the economic mechanism.

(d) **[7 pts]** *Risky savings.* Suppose instead that income is certain, $Y_1 = \bar{Y}$, but the gross return on saving is

$$R^g_1 = \bar{R} \exp\left(\sigma Z - \frac{\sigma^2}{2}\right), \qquad Z \sim N(0,1), \quad \sigma > 0.$$

Show that $\mathbb{E}_0[R^g_1] = \bar{R}$. Using the Euler equation, determine whether return uncertainty increases or decreases optimal saving relative to part (b). Justify your answer and explain why the effect differs from income uncertainty.

(e) **[4 pts]** Consider the risky-return environment of part (d), but suppose the household receives no future labour income, $Y_1 = 0$. Derive optimal saving explicitly. Does the distribution of R^g_1 affect optimal saving? Explain briefly.