Empirical Asset Pricing: S02
Cross Section of Stock Returns

Amedeo Andriollo

Finance Department (DRM)
Université Paris Dauphine - PSL.

Structure of the course

- Email: amedeo.andriollo@dauphine.psl.eu
- Grading follows a 30-70 rule: 70% final exam, 30% project/homework (3rd and 5th sessions).
- Dense slides. References at the end.
- HW: if we do not answer in class, those are homework: try to answer them.
- Office hours: feel "free" to (DO) come.
- If you spot any typos/mistake, please let me know: slides are updated regularly.

⊵ More updates will follow.

A Quick Warm Up: Previous Session

- Excess returns and prices: $R_{i,t+1} = (D_{i,t+1} + P_{i,t+1})/P_{i,t} - 1 - R_{f,t+1}$
- SDF: for asset i, $1 = \mathbb{E}_t[M_{t+1}(1 + R_{f,t+1} + R_{i,t+1})]$, or: $\mathbb{E}_t[M_{t+1}R_{i,t+1}] = 0$.
- Using the def. of covariance (**HW:** def. of $R_{f,t}$?), we write the beta-pricing model:

$$\mathbb{E}_t[R_{i,t+1}] = (1 + R_{f,t+1})\text{Cov}_t(-M_{t+1}, R_{i,t+1})$$

$$\mathbb{E}_t[R_{i,t+1}] = \beta_i \lambda_t = \left\{ \frac{\text{Cov}_t(-M_{t+1}, R_{i,t+1})}{\text{Var}_t(M_{t+1})} \right\} \{(1 + R_{f,t+1})\text{Var}_t(M_{t+1})\}$$

$\hookrightarrow$ risky assets (stock, bond, ..) should correlated with the SDF, such that expected excess return are prop. to the beta coefficient in the regression of return onto SDF, β_i. The constant of proportionality, **common** to the cross-section of assets, is the price of risk at time t, λ_t.

- What if $M_t = a - bR_{i,t}$?
  Then: $\mathbb{E}_t[R_{i,t+1}] = b(1 + R_{f,t+1})b^{2-2}\text{Var}_t(R_{i,t+1}) = b(1 + R_{f,t+1})\text{Var}_t(R_{i,t+1})$
  HW: What is a pinning down?

Portfolios and Tangency Portfolio (1)

- Usually we employ portfolios, that are linear combination of returns: $\sum_{i,t} \omega_i R_{i,t} = \omega_t' R_t$.
  $\hookrightarrow$ investors wish to invest in multiple assets/claims, possibly hedging diverse risks.

- Usually we have: $\sum_i \omega_i = 1$. What if weights are $>> 1$? Leverage (e.g., the 130-30 strategy: short underperf. stocks and reinvest the collected capital into high(er)-perf. stocks).
  $\hookrightarrow$ Zero-investment strategies: long vs. short portfolio, identified by some trading rule.

- SDF pricing requires no arbitrage:
  $\hookrightarrow$ there exists a mean-variance frontier, with tangency portfolio $R^\star$, $\text{Var}_t[R^\star_{t+1}] > 0$.

- What if there is arbitrage? (Asymptotically,) the investors get rich with vanishing risk
  $\implies \text{Var}_t(R^\star_{t+1}) \to 0$, thus we can't get the betas (division by zero).

Portfolios and Tangency Portfolio (2)

- For any rescaling $a > 0$, the portfolio with weights $\omega_t(a) = a \times \mathsf{Var}_t(R_{t+1})^{-1}\mathbb{E}_t[R_{t+1}]$ is (conditional) Mean-Variance (MV) efficient.

- For any rescaling $a' > 0$, the MV portfolio $R^\star_{t+1}(a') = \omega_t(a')R_{t+1}$ leads to the beta-pricing:

$$\mathbb{E}_t[R_{i,t+1}] = \left(\mathsf{Cov}_t(R^\star_{t+1}, R_{i,t+1})/\mathsf{Var}_t(R^\star_{t+1})\right)\mathbb{E}_t[R^\star_{t+1}] = \beta_{i,t}\lambda_t$$

  (Proof in the next slide).

- At equilibrium, prices are affected by a, however the beta proportionality holds.

- The tangency portfolio, $R^\star_{t+1}(a^\star)$, corresponds to a choice of a: $\sum_i \omega_{i,t}(a^\star_t) = \mathbf{1}'\omega_t = 1$.

Beta-pricing and SDF

- From SDF to beta-pricing? By optimality of the mean-variance frontier: (given $a > 0$) $\max_{\omega_t} \mathbb{E}_t(\omega_t' R_{t+1}) - \frac{a}{2}\text{Var}_t(\omega_t' R_{t+1})$, with (**HW**:) FOC: $\mathbb{E}_t(R^\star_{t+1}) = a \times \text{Var}_t(R^\star_{t+1})\omega_t$,

$$\mathbb{E}_t(R_{i,t}) = a \times \beta_{i,t}\text{Var}_t(R^\star_{t+1}), \quad \beta_{i,t} = \frac{\text{Cov}_t(R_{i,t+1}, R^\star_{t+1})}{\text{Var}_t(R^\star_{t+1})}, \quad \text{for } i \text{ asset}$$

  where $\beta_{i,t} = 1$ implies: $\lambda_t := \mathbb{E}_t[R^\star_{i,t+1}] = a\text{Var}_t(R^\star_{t+1})$, and so: $\mathbb{E}_t(R_{i,t}) = \beta_{i,t}\lambda_t$.
- From beta-pricing to SDF? Yes: if exists a portfolio with excess returns $R^\star_{t+1}$ such that is beta-pricing all i assets $\implies$ the SDF is defined as:

$$M_{t+1} = \frac{1}{1 + R_{f,t+1}}\left(1 - (R^\star_{t+1} - \mathbb{E}_t[R^\star_{t+1}])\frac{E_t[R^\star_{t+1}]}{\text{Var}_t[R^\star_{t+1}]}\right)$$

- Notice that: 1) $E_t[R^\star_{t+1}]/\text{Var}_t[R^\star_{t+1}] = \beta_{i,t}\lambda_t/\text{Cov}_t[R^\star_{t+1}, R_{i,t+1}]$, 2) (**HW**:) By construction: $\mathbb{E}_t[M_{t+1}R_{i,t+1}] = 0$, where $\mathbb{E}_t[M_{t+1}] = 1/(1 + R_{f,t+1})$.

Beta-pricing and Mimicking Portfolios

- Mimicking/Tracking Portfolios: projection/regression of negative SDF onto returns.
  ↪ It mimicks the SDF co-movements, and makes it tradable (Breeden, 1979).

that is: $-M_{t+1} = c_t + b_t' R_{t+1} + u_{t+1}$, which leads to: **(HW:)**
$b_t = \omega_t(\tilde{a}_t), \quad c_t = -\tilde{a}_t - \omega_t(\tilde{a}_t)'\mathbb{E}_t[R_{t+1}], \quad \tilde{a}_t = 1/(1 + R_{f,t+1})$.

- The mimicking portfolio is proportional to the tangency portfolio: $\omega_t(\tilde{a}_t) \propto \omega_t(a_t^\star)$, and has beta-pricing properties (due mean-variance efficiency).
- Any SDF can be written as:

$$M_{t+1} = \frac{1}{1 + R_{f,t+1}} \left(1 - \mathbb{E}_t[R_{t+1}]'\text{Var}_t(R_{t+1})^{-1}(R_{t+1} - \mathbb{E}_t[R_{t+1}])\right) + e_{t+1}$$

↪ The min variance SDF is full projection: $e_{t+1} = 0$ (complete mkt $\implies$ unique SDF).

- Notice that M_{t+1} is made of four parts:
  i) $1/(1 + R_{f,t+1})$, ii) e_{t+1}, iii) $\mathbb{E}_t[R_{t+1}]'\text{Var}_t(R_{t+1})^{-1}\mathbb{E}_t[R_{t+1}]/(1 + R_{f,t+1})$,

  and iv) $-\mathbb{E}_t[R_{t+1}]'\text{Var}_t(R_{t+1})^{-1}R_{t+1}/(1 + R_{f,t+1}) = -\omega_t(\tilde{a}_t)R_{t+1}$ (the beta-pricing part)

Sharpe Ratio and MV efficiency

- MV efficiency connects to the Sharpe Ratio: $SR_t(R_{i,t+1}) = \mathbb{E}_t[R_{i,t+1}]/\sigma_t(R_{i,t+1})$
- Using the identity between correlation and covariance, we can write from the beta-pricing: (**HW**:) $|\mathbb{E}_t[R_{t+1}]|/\sigma_t(R_{t+1}) \leq \sigma_t(M_{t+1})/\mathbb{E}_t[M_{t+1}]$, with $\mathbb{E}_t[M_{t+1}] = 1/(1 + R_{f,t+1})$
  $\hookrightarrow$ restrictions on the SDFs that price a given set of returns (Hansen and Jagannathan, 1991).
- How to read:
  - Sharpe ratio of any portfolio is bounded above by the volatility of the SDF.
  - the SR of tangency portfolio is the highest and equal to:

$$\frac{\mathbb{E}_t[R^\star_{t+1}]}{\sigma_t(R^\star_{t+1})} = \frac{b_t'\mathbb{E}_t[R_{t+1}]}{\sqrt{b_t'\text{Var}_t[R_{t+1}]b_t}} = \sqrt{\mathbb{E}_t[R_{t+1}]'\text{Var}_t[R_{t+1}]^{-1}\mathbb{E}_t[R_{t+1}]}$$

Conditional and Unconditional MV efficiency (1)

- Conditional MV efficiency set the weights to be $a-$proportional to $\text{Var}_t(R_{t+1})^{-1}\mathbb{E}_t[R_{t+1}]$. Uncond. MV efficient: $\breve{a}_t = (1 + \mathbb{E}_t[R_{t+1}]'\text{Var}_t[R_{t+1}]^{-1}\mathbb{E}_t[R_{t+1}])^{-1}$.

$\hookrightarrow$ the Uncond. is constructed from the Cond. using a leverage that leads to max uncond. SR. the Uncond. MV efficient portfolio $\Longrightarrow$ Uncond. beta-pricing.

- For any w_t-portfolio, the conditional beta-pricing holds when: $\mathbb{E}_t[w_t' R_{t+1}] = \beta_t \lambda_t, \quad \beta_t = \text{Cov}_t(w_t' R_{t+1}, R^\star_{t+1})/\text{Var}_t(R^\star_{t+1})$

- The Uncond. MV efficient (UMVE) portfolio holds conditionally and also leads also to: $\mathbb{E}[\omega_t(\breve{a}_t)' R_{t+1}] = \beta\lambda, \quad \beta = \text{Cov}(\omega_t(\breve{a}_t)' R_{t+1}, R^\star_{t+1})/\text{Var}(R^\star_{t+1})$

Conditional and Unconditional MV efficiency (2)

↪ we have the generic Conditional and Unconditional excess return representations:

$$\mathbb{E}_t[w_t R_{t+1}] = \beta_t R^{\star}_{t+1} + \delta_t' U_{t+1} + \varepsilon_{t+1}, \qquad \mathbb{E}_t[U_{t+1}] = \mathbb{E}_t[\varepsilon_{t+1}] = 0$$
$$\mathbb{E}[w_t R_{t+1}] = \beta R^{\star}_{t+1} + \delta' U_{t+1} + e_{t+1}, \qquad \mathbb{E}[U_{t+1}] = \mathbb{E}[e_{t+1}] = 0$$

where the ϵ, e stand for the idiosyn. noise, and $\{U_t\}$ capture common unpriced components
↪ U's as hedge portfolios: zero risk premium and removing them increseas the SR.

- In general: $\mathbb{E}[\mathbb{E}_t[w_t' R_{t+1}]] \neq \mathbb{E}[\beta_t]\mathbb{E}[\lambda_t]$. For UMVE holds, since UMVE $\implies$ CMVE.

$$\mathbb{E}[w_t' R_{t+1}] = \mathbb{E}[\beta_t]\mathbb{E}[\lambda_t] + \text{Cov}(\beta_t, \lambda_t) = \beta\mathbb{E}[\lambda_t] + (\mathbb{E}[\beta_t] - \beta)\mathbb{E}[\lambda_t] + \text{Cov}(\beta_t, \lambda_t)$$

where the first term is: i) the unconditional beta-pricing,
ii) "volatility timing" and iii) "equity premium timing" effects.
(pg.60-1, Campbell, 2017).

CAPM (1)

- If all investors want to max out their SR, by choosing a combination of tangency and risk-free portfolios, we have: $R_t^{MKT} = R_t^{\star}$.
- We expected the market portfolio to price the assets in (unconditional) beta-pricing terms:

$$\mathbb{E}[R_{i,t}] = \beta_i \lambda, \quad \beta_i = \text{Cov}(R_{i,t}, R_t^{MKT}), \quad \lambda = \mathbb{E}[R_t^{MKT}]$$

  where λ is the MKT risk premium. **HW:** Write the SDF.
- What does the model say? MKT portfolio lies on the (unconditional) MV frontier.

- What can we **not** test via CAPM?
  - Rational investors $\implies$ existence of SDF/beta-pricing. (Reverse does not hold).

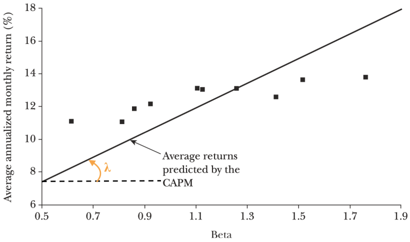
  $\hookrightarrow$ Ex ante expected retruns and betas correspond to the equilibrium (sample) distributions?
  - (Conditional) CAPM: i) betas change over firm's life-cycle, ii) mkt risk over the business cycles.
  - Do we observe the market portfolio, R^{MKT}?

    $\hookrightarrow$ rejection of CAPM may depend on the measurement error in mkt returns (e.g., nontradables).

CAPM (2)

- Deviations from beta-pricing are the $\alpha's$: $\alpha_i = \mathbb{E}[R_{i,t}] - \beta_i \lambda$
- The Security Market Line:

Figure 2
Average Annualized Monthly Return versus Beta for Value Weight Portfolios Formed on Prior Beta, 1928–2003



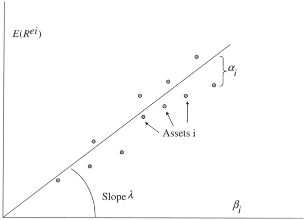


Figure 12.1. *Cross-sectional regression.*

On the left: the figure plots ten beta-sorted portfolio's average return against their postranking beta, estimated by regressing their monthly returns for 1928–2003 on the return on the CRSP value-weight portfolio of U.S. common stocks (Fama and French, 2004). On the right, Cochrane (2009): chp 12.

Testing the CAPM (1)

- Suppose we have a MKT proxy/factor and $k = 1, .., N$ assets. Estimating the CAPM: time-series reg of excess returns on excess MKT returns.

$$r_{k,t} = \alpha_k + \beta_k r_{MKT,t} + \epsilon_{k,t}$$

$\hookrightarrow$ Estimates: $\{\widehat{\alpha}_k, \widehat{\beta}_k, \widehat{\epsilon}_{k,t}\}$ via OLS for each asset $k = 1, ..., N$ (seemingly unrelated reg.):

$\widehat{\beta}_k = \widehat{\text{Cov}}(r_{k,t}, r_{MKT,t})/\widehat{\sigma}^2_{MKT}$, $\quad \widehat{\alpha}_k = \bar{r}_k - \widehat{\beta}_k \bar{r}_{MKT}$, $\quad \widehat{\epsilon}_{k,t} = r_{k,t} - \widehat{\alpha}_k - \widehat{\beta}_k r_{MKT,t}$

with the sample means: $\widehat{\lambda}_{MKT} = \bar{r}_k = T^{-1} \sum_{t=1}^{T} r_{k,t}$, $\quad \bar{r}_{MKT} = T^{-1} \sum_{t=1}^{T} r_{MKT,t}$,
and the sample variance: $\widehat{\sigma}^2_{MKT} = (T-1)^{-1} \sum_{t=1}^{T} (r_{MKT,t} - \bar{r}_k)$

- Testing whether the CAPM "work": is the MKT MVE?
  - Looking at the measure of goodness of fit: $\widehat{R}^2 = 1 - \left(\sum_{t=1}^{T} \widehat{\epsilon}^2_{k,t}\right) / \left(\sum_{t=1}^{T} (r_{k,t} - \bar{r}_k)^2\right)$

  $\hookrightarrow$ Low R squared would implies there is a lot of idiosyncratic risk.

Testing the CAPM (2)

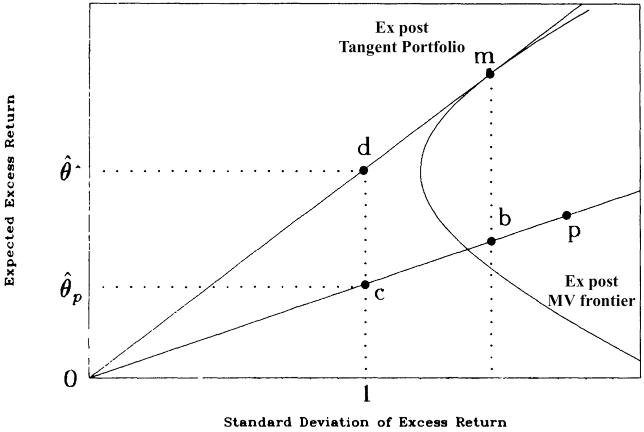
- Rather, CAPM predicts that "differences in expected return across securities and portfolios are entirely explained by differences in market beta": all pricing errors ($\alpha' s$) are jointly $= 0$.

$$\mathcal{H}_0 : \alpha_k = 0, \quad \text{for all } k = 1, ..., N$$

- Define: 1) The $N \times 1$ vector $\widehat{\alpha} = (\widehat{\alpha}_1, ..., \widehat{\alpha}_N)'$, and $\widehat{\epsilon}_t = (\widehat{\epsilon}_{1,t}, ..., \widehat{\epsilon}_{1,N})'$.
  2) The $N \times N$ Variance-Covariance matrix of the error terms $\widehat{\Sigma}_e$: $T^{-1} \sum_{t=1}^{T} \epsilon_t \epsilon_t'$.
  Note that:
  - The ex post/realized risk premium of MKT is its historical average: $\bar{r}_{MKT}$.
  - The sample/realized squared Sharpe Ratio of MKT: $\widehat{SR}^2_{MKT} = \bar{r}^2_{MKT} / \widehat{\sigma}^2_{MKT}$.

- Test statistics:
  - Wald test: $W = T \left(1 + \widehat{SR}^2_{MKT}\right)^{-1} \widehat{\alpha}' \widehat{\Sigma}_\epsilon^{-1} \widehat{\alpha} \xrightarrow[\text{under } \mathcal{H}_0]{d} \chi^2_N$
  - Gibbons et al. (1989)'s GRS (finite-sample) test: $GRS = \dfrac{T - N - 1}{NT} W \xrightarrow[\text{under} \mathcal{H}_0]{d} F_{N, T-N-1}$

Testing the CAPM (3)



[Figure 1a, Gibbons et al. (1989)]

- Assumptions on conditional returns ($\epsilon' s$): normality, serial uncorrelation and homoskedasticity.
- Is the MKT portfolio on the *ex ante* MV frontier? Rather: *ex post* MV frontier and tangency portfolio: $N+1$ assets.
- In the figure: 1) Sample SR of MKT: $\widehat{SR}_{MKT} = \widehat{\theta}_p$, 2) Max SR per unit of risk: $\widehat{\theta}^{\star}$.
- We have: $\dfrac{W}{T} = \dfrac{\widehat{\alpha}'\widehat{\Sigma}_{\epsilon}^{-1}\widehat{\alpha}}{1+\widehat{SR}^2_{MKT}} = \dfrac{\overline{Od}}{\overline{Oc}}$

Testing the CAPM (4)

- Wald/GRS tests use the Time-Series (TS), not leveraging the Cross-Sectional (XS) info.

↪ Linear reg. of $\mathbb{E}[R_{i,t}]$ onto β_i: the slope is λ, and the residuals are zero on avg.:

1. TS regression to get the asset-level beta exposures: $\{\widehat{\beta}_k\}$. [First pass]
2. XS reg. of the average excess returns, $\{\bar{r}_i\}$, on the betas, $\{\widehat{\beta}_i\}$ [Second pass]:

$$\bar{r}_i = \gamma_0 + \widehat{\beta}_i \gamma + u_i, \qquad \mathbb{E}[u_i] = 0$$

3. Testing CAPM: $\mathcal{H}_0^{(1)} : \gamma_0 = 0 (= N^{-1} \sum_{k=1}^{N} \alpha_k)$ and $\mathcal{H}_0^{(2)} : \gamma = \bar{r}_{MKT}$

- Rejection $\implies$ MKT is not on the MV frontier.
  - Some risk not spanned entirely by MKT. Look at the residuals of the second step, the $u's$: *"Do assets (firms) for which the pricing error is positive have something in common?"*
  - "Idiosyncratic" risk that is priced (= nonzero risk premium)

  ↪ Multifactor model: more than one risk dimension

$$r_{k,t} = \alpha_k + \beta_{k,MKT} R_t^{MKT} + \sum_{l=1} \beta_{k,l} f_{l,t} + \epsilon_{k,t}$$

On Fama MacBeth's approach (1)

- Original Fama and MacBeth (1973): conditional model with $\{\lambda_t\}$

  ↪ Having asset-level betas, XS reg. period-by-period of excess returns on estimated betas. Then avg. over time for the unconditional quantities.

  - Conditional vs. Unconditional? Which assets? Individual/Portfolios?

- In general, for any $N \times 1$ vector of (individual) asset, we expect a conditional model to hold:

$$r_{t+1} = \beta_t v_{t+1} + \beta_t \mathbb{E}_t[R^\star_{t+1}] + u_{t+1}$$

↪ Using instruments/managed portfolios, so that firm-level betas are (linearly) prop. to $M << N$ firm-level characteristics: $\beta_t = c_t \beta$, where c_t is $N \times M$ matrix of characteristics.

- From Conditional to Unconditional using portfolios (from N to M assets):

$$(c_t' c_t)^{-1} c_t' r_{t+1} = \beta(v_{t+1} + \mathbb{E}_t[R^\star_{t+1}]) + (c_t' c_t)^{-1} c_t' u_{t+1}$$

  (**HW**: Is there a trade-off?)

- Portfolios can also be used to average noisy betas (individual stock are too volatile).

On Fama MacBeth's approach (2)

- Fama and MacBeth (1973)'s intuition about the linear relationship between stock-level characteristics and returns: $R_{i,t} = c_t + \theta_t X_{i,t} + a_{i,t}$
  - The slope, $\widehat{\theta}_t = \widehat{\text{Cov}}(r_{i,t}, X_{i,t}) / \widehat{\text{Var}}(X_{i,t})$, is the payoff at time t on a long-short portfolio that invests in each stock proportional to its characteristic $X_{i,t}$, financing long with short, scaling by sum of squared characteristics.
  - (**HW**: What is the interpretation of $\widehat{c}_t$?).

- It might be nonlinear: (good) old tables of alphas, and now ML algorithms.
  However, monotonicity: "In practice, a relationship between a characteristic and average return is regarded with particular suspicion if it is non linear enough to be non-monotononic" (Campbell, 2017).

- What are the difference between methods? Shall we look at the XS or TS?
  Suppose the panel data regression: $y_{i,t} = x_{i,t}\beta + \varepsilon_{i,t}, \quad i = 1, ..., N, \quad t = 1, ..., T.$

On Fama MacBeth's approach (3)

- Cochrane (2009): if the $x_{i,t}$ variables do not vary over time ($x_{i,t} = x_i$), and if the errors are uncorrelated over time, then:
  i) pooled OLS, ii) pure XS OLS, and iii) Fama-MacBeth estimates of betas are identical.
  Stacked notation: $Y = X\beta + \varepsilon$, $Y = (y_1, ..., y_T)'$, $X = (x, ..., x)'$, $\varepsilon = (\varepsilon_1, ..., \varepsilon_T)'$
  i) Pooled OSL.
  $\widehat{\beta}_{OLS} = (X'X)^{-1}X'Y = (x'x)^{-1}x'\bar{y}$. $\text{Var}(\widehat{\beta}_{OLS}) = (X'X)^{-1}X'\mathbb{E}[\varepsilon\varepsilon']X(X'X)^{-1}$.
  (**HW**: $\text{Var}(\widehat{\varepsilon}) = ?$)
  ii) Pure XS: $\bar{y} = \bar{x}\beta + \bar{\varepsilon} = x\beta + \bar{\varepsilon}$, since $\bar{x} = x$. $\widehat{\beta}_{XS} = (x'x)^{-1}x'\bar{y}$.
  When errors are i.i.d. over time: $\text{Var}(\bar{\varepsilon}) = T^{-1}\Sigma$, where: $\mathbb{E}[\varepsilon\varepsilon'] = \text{diag}(\Sigma)$.
  Then: $\text{Var}(\widehat{\beta}_{XS}) = \frac{1}{T}(x'x)^{-1}x'\Sigma x(x'x)^{-1} = \text{Var}(\widehat{\beta}_{OLS})$, since $X'X = Tx'x$.
  iii) Fama-MacBeth: first step, $\widehat{\beta}_t = (x'x)^{-1}x'y_t$; second step, TS avg. of XS betas:
  $\widehat{\beta}_{FM} = \overline{\widehat{\beta}_t} = T^{-1}\sum_{t=1}^{T}\widehat{\beta}_t = (x'x)^{-1}x'\bar{y}$.
  Variance of the estimator is: $\text{Var}(\widehat{\beta}_{FM}) = \frac{1}{T}(x'x)^{-1}x'\text{Var}(y_t)x(x'x)^{-1}$.
  $\hookrightarrow$ when the linear model: $y_t = x\beta_{FM} + \varepsilon_t$, then: $\text{Var}(y_t) = \widehat{\Sigma}$

Factor model (1)

- Suppose the $N \times 1$ excess returns follows the (multi-)factor model:

$$r_t = \alpha + \beta f_t + \epsilon_t, \quad \mathbb{E}[\epsilon] = 0, \quad \mathbb{E}[f_t \epsilon_t] = 0$$

- We have:
    - Expected returns are: $\mathbb{E}[r_t] = \alpha + \beta \mathbb{E}[f_t]$.
    - Variance is: $\Sigma_r := \text{Var}(r_t) = \beta \text{Var}(f_t) \beta' + \text{Var}(\epsilon_t) = \beta \Sigma_f \beta' + \Sigma_\epsilon$.
    - The betas are: $\beta = \text{Cov}(r_t, f_t)(\text{Var}(f_t))^{-1} = \Sigma_{rf} \Sigma_f^{-1}$
    - If returns and factors are jointly normal, then the conditional mean: $\mathbb{E}[r_t | f_t] = \alpha + \beta f_t$
- We say that the factor prices (unconditionally) all assets when: $\mathbb{E}[r_t] = \beta \gamma$, so:

$$r_t = \beta \gamma + \beta (f_t - \mathbb{E}[f_t]) + \varepsilon, \qquad \alpha = \beta(\gamma - \mathbb{E}[f_t])$$

- When the (observable) factor is:
    - Tradable (it is a portfolio): $\gamma = \mathbb{E}[f_t] \implies$ TS testing of alphas equal to zero. Same reasoning holds in multivariate context.
    - Non-tradable ($f_t \propto m_t$) $\implies$ XS testing.

Factor model (2)

- Suppose the factors are tradable, then the full market is spanned by: $z_t = (f'_t, r'_t,)'$.
- The vector of expected returns: $\mu = (\mu'_f, (\alpha + \beta\mu_f)')'$
- The matrix of variance-covariance: $V = \begin{pmatrix} \Sigma_f & \Sigma_f \beta' \\ \beta\Sigma_f & \beta\Sigma_f\beta' + \Sigma_\epsilon \end{pmatrix}$
- Its inverse is: $V^{-1} = \begin{pmatrix} \Sigma_f^{-1} + \beta'\Sigma_\epsilon^{-1}\beta & -\beta'\Sigma_\epsilon^{-1} \\ -\Sigma_\epsilon^{-1}\beta & \Sigma_\epsilon \end{pmatrix}$
- For a given target e, the min variance frontier is defined: $\min q'Vq$, s.t. $q'\mu = e$. setting the Lagrangian: $L = q'Vq + 2\pi(e - q'\mu)$, leads to:
  - The frontier portfolio is: $q = \pi V^{-1}\mu$.
  - Squared SR of the frontier portfolio is: $\mu' V^{-1}\mu = \mu'_f \Sigma_f^{-1}\mu_f + \alpha'\Sigma_\epsilon\alpha$
- If the factors are (uncond.) beta-pricing the assets, then:
  1) they should be on the min variance SDF frontier, 2) they attain the max SR
  $\hookrightarrow \alpha = 0 \implies \mu' V^{-1}\mu = \mu'_f \Sigma_f^{-1}\mu_f$.

On Fama MacBeth's approach (4)

- Suppose the $N \times 1$ excess returns follows the (multi-)factor model:

$$r_t = \alpha + \beta f_t + \epsilon_t, \quad \mathbb{E}[\epsilon] = 0, \quad \mathbb{E}[f_t \epsilon_t] = 0$$

- The XS (OLS) regression uses the betas as regressors:
  - XS in vector notation: $\bar{r} = \beta\gamma + \gamma_0 + u = \beta\gamma + \alpha, \quad \mathbb{E}[\alpha] = 0$

  $\hookrightarrow$ the XS estimates are: $\widehat{\gamma} = (\beta'\beta)^{-1}\beta'\bar{r}$, and $\widehat{\alpha} = \bar{r} - \beta\widehat{\gamma} = (I - \beta(\beta'\beta)^{-1}\beta')\bar{r}$

  - The variances are:
    $\text{Var}(\widehat{\gamma}) = T^{-1}(\beta'\beta)^{-1}\beta'(\beta\Sigma_f\beta' + \Sigma_\varepsilon)\beta(\beta'\beta)^{-1} = T^{-1}\Sigma_f + T^{-1}(\beta'\beta)^{-1}\beta'\Sigma_\varepsilon\beta(\beta'\beta)^{-1}$
    $\text{Var}(\widehat{\alpha}) = T^{-1}(I - \beta(\beta'\beta)^{-1}\beta')(\beta\Sigma_f\beta' + \Sigma_\varepsilon)(I - \beta(\beta'\beta)^{-1}\beta')'$.

  $\hookrightarrow$ since $\beta - \beta(\beta'\beta)^{-1}\beta'\beta = 0$, we have: $\text{Var}(\widehat{\alpha}) = T^{-1}(I - \beta(\beta'\beta)^{-1}\beta')\Sigma_\varepsilon(I - \beta(\beta'\beta)^{-1}\beta')'$

On Fama MacBeth's approach (5)

- If you have cross-correlation between residuals: GLS should be used.
- The XS GLS regression considers the betas as regressors but correlated:
  - the XS estimates are then: (Cochrane (2009): chp 12.)
    $\widehat{\gamma}_{GLS} = (\beta' \Sigma_\varepsilon^{-1} \beta)^{-1} \beta' \Sigma_\varepsilon^{-1} \bar{r}$,
    $\widehat{\alpha}_{GLS} = (I - \beta(\beta' \Sigma_\varepsilon^{-1} \beta)^{-1} \beta' \Sigma_\varepsilon^{-1}) \bar{r}$
- The betas are not observed but (using the TS) estimated (Shanken, 1992):

$$\text{Var}(\widehat{\gamma}) = T^{-1} \Sigma_f + T^{-1} (\beta' \beta)^{-1} \beta' \Sigma_\varepsilon \beta (\beta' \beta)^{-1} (1 + \gamma' \Sigma_f^{-1} \gamma)$$

- The correction term is proportional to the squared SR of the factors.

↪ How important is the correction term? Looking at CAPM:
  - Annual data: squared SR of MKT is around 0.25.
  - Monthly data: squared SR of MKT is around 0.02.

Thanks for your attention! See you next Monday!

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